

NONSUPERSYMMETRIC INTERSECTING BRANES IN SUPERGRAVITY

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Abstract

In this doctoral thesis a model of many orthogonally commonly intersecting delocalized branes with neither harmonic gauge nor any other extra conditions is discussed. Further a method of solving equations of motion of the model is given. It is proved that the model reduces to the so called Toda-like system which is solvable at least in several cases relevant for realistic brane configurations. The solutions generally can break supersymmetry. Examples of the solutions are given and some their properties are considered in more detail. Especially the presence and interpretation of singularities is discussed and the relation between energy and charge density of the solution. A certain duality in the space of solutions is described connecting two seemingly different elements of the space. It is shown that the solution dual to the supersymmetric one breaks supersymmetry, but it still possesses some features usually attributed only to solutions preserving supersymmetry. In particular for the dual solution equality between energy and charge density holds.

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1 Introduction

Branes are one of the most interesting topics that appeared in the theoretical physics of elementary particles in the last years. Treated at the beginning as just some special solutions of supergravity models they slowly gathered importance to the point that now they are treated as hypothetically more fundamental than strings. In superstring theories branes are identified with sources of Ramond-Ramond charges. These branes, so called D-branes, are equivalently described as hypersurfaces where ends of open strings are attached. Consequently, the standard type I string theory with freely propagating open strings can be understood as a theory on a ten dimensional brane worldvolume. But there is also another sort of branes – NS-branes and a special representative of the category is a fundamental string serving as an elementary object of the whole string theory. This fact allows to make a conjecture that there can be a theory – a generalization of string theory – where not strings but branes are the fundamental objects. The hypothesis suggests that this tentative M-theory could be realized as a quantum theory of membranes but the explicit realization of this idea has not yet been found.

But even restricted to string theory the role of branes is extraordinarily important. Branes belong to the spectrum of the theory, so the knowledge of their properties is necessary for the better understanding of string theory. Discovery of branes significantly enlarged the number of known states in the theory, and since they don't have a string interpretation their presence in the spectrum sheds new light on the whole theory. Several classes of states can be distinguished depending on their features. For example BPS states, which saturate so called Bogomolny bound and preserve at least part of supersymmetry. The remaining non-BPS states fall into two main categories: stable and unstable where the last one contains tachyons. Non-BPS stable states are especially interesting because the physics we see (which is obviously nonsupersymmetric) is probably the low energy limit of such a state with (softly) broken supersymmetry.

A fact worth to note at this moment is that branes are in general objects of certain dimensionality living in a higher dimensional theory. So it is tempting to interpret the four dimensional Universe we observe as the brane immersed in ten dimensional spacetime of string theory.

Since branes can be treated as an extension of some already well known (at least theoretically) objects as magnetic monopoles or fundamental strings, they allow to look at some previously discovered facts from a wider perspective and find connections among them. For example theorems telling that classical superstrings are permitted only for $D = 3, 4, 6, 10$ and the maximal dimension for supergravity is eleven are consequences of a rule determining how many dimensional superbranes are possible to introduce in a given spacetime. The idea of branes provides also a method for studying Yang-Mills theories in diverse dimensions. This comes from a fact that a theory induced on the brane worldvolume by the surrounding string theory is in general gauge invariant. So one can understand properties of a given Yang-Mills theory in connection with properties of the branes. In particular BPS branes give a description of super-Yang-Mills theories, while non-BPS branes – gauge theories with broken supersymmetry.

Branes can also be regarded from the purely classical point of view as solutions of field equations in supergravity or more generally in systems with gravity coupled to antisymmetric tensors and other fields. This method of description of branes is very important because it gives some exact information which cannot be obtained with the use of perturbative methods peculiar for string theory. The main subject of this work is dedicated to this part of the theory of branes, especially to such brane solutions of supergravity equations of motion which can break supersymmetry. We aim in this work to find and analyze new brane solutions that are in general nonsupersymmetric and which describe many intersecting branes supported either by the same type of field or by different fields.

Before we go over to the main part of the work we present a short explanation what branes are and why they are interesting. In chapter 2 a path is presented which leads from the Standard Model and General Relativity – the theories extremely well describing physics from the smallest to the biggest scales experimentally testable at this moment – through Grand Unification Theories, models with extra dimensions, supersymmetry and supergravity to bosonic strings, superstrings and finally

M-theory, currently thought to be the best candidate for the unified theory of all interactions.

Various kinds of branes are introduced afterwards, in chapter 3. First of all a possibility of constructing quantum theories based on branes as its fundamental objects (instead of point particles or strings) is examined. A result of so called brane scan is presented, which tells for what combinations of a spacetime and the brane worldvolume dimensions it is possible to introduce a supersymmetric brane. The discussion however ends with the conclusion that such a theory as we understand it now cannot be consistently quantized. Therefore we gather information about branes by treating them as sources of fields of antisymmetric forms and we obtain generalization of the famous Dirac quantization rule for magnetic monopoles. Next we discuss branes appearing in M- and string theories in particular D-branes, NS-branes and M-branes and give arguments that branes are the necessary part of the theories. We present the classification of the brane states with respect to the saturation of the Bogomolny bound and show some examples of non-BPS unstable and stable states. We check also how the various kinds of branes behave under string dualities or dimensional reductions. The observations allow us to conjecture that all the branes can be various manifestations of only one class of more fundamental objects.

Finally in chapter 4 supergravity description of branes is given starting with detailed discussion of a simple but very instructive example – single component brane solution in the harmonic gauge. Some generalizations of the example including black branes and systems with many intersecting branes are also presented.

The last chapter consists of a construction of a model of many orthogonally commonly intersecting delocalized branes with neither harmonic gauge nor any other extra conditions. Further a method of solving equations of motion of the model is given. It is proved that the model reduces to the so called Toda-like system (after adequate redefinition of radial coordinate: $r \rightarrow \vartheta(r)$, where $\vartheta(r)$ is in general not a harmonic function in flat space). The system is solvable at least in several cases relevant for realistic brane configurations. The solutions generally can break supersymmetry and the supersymmetric solutions are generally distinguished by their very specific properties. Examples of the solutions are given and some their properties are considered in more detail. Especially the presence and interpretation of singularities is discussed and the relation between energy and charge density of the solution. A certain duality in the space of solutions is described connecting two seemingly different elements of the space. It is shown that the solution dual to the supersymmetric one breaks supersymmetry, but it still possesses some features usually attributed only to solutions preserving supersymmetry. In particular for the dual solution equality between energy and charge density holds.

2 From the Standard Model and General Relativity to superstrings.

Nowadays physicists, especially those interested in theories of elementary particles sometimes seem to be modern incarnations of the legendary King Arthur's knights. Similarly to the ancient warriors, they spend a lot of time and devote most of their efforts to achieve one goal for which there is even no known proof that it really exists. The goal however is not the Holy Graal. The physicists are searching for something even more miraculous – the Theory of Everything, how is often called the hypothetical unified theory of all interactions. The road expected to lead to the goal is very difficult, winding and with many branches ending as blind alleys. Fortunately there are no dragons or other beasts lurking for inadvertent travellers. But striding the way one can find several other creatures lying at the shoulder of the road as a milestones of scientific progress. Creatures, which in many cases are still not fully domesticated and which at any time can make a surprise for the explorer. The list of the creatures is long and contains grand unification theories, supersymmetries and supergravities, extra dimensions, strings and superstrings and one of the latest discovery for which this work is dedicated – branes. Let us look at some of their features more closely.

2.1 General Relativity and the Standard Model.

In modern physics we know four fundamental interactions described by two completely distinct theories. One of them is Einstein's General Relativity [1, 2, 3, 4] describing classically gravitational forces. The other is the Standard Model [5, 6, 7, 8], quantum field theory of electromagnetic, weak and strong interactions.

The General Relativity Theory is based on Equivalence Principle i.e. a postulate of an invariance under general coordinate transformations (in the usual formulation) or the local $ISO(3, 1)$ or Poincaré symmetry in a tangent space (in so called Einstein–Cartan formulation). The theory describes particles of spin 2 (gravitons) and has a very elegant and simple structure because its lagrangian is given just by $\frac{1}{2\kappa}R[g] + \lambda$. One of the most significant achievements of the theory is a direct link between physics and geometry because the kinetic term for gravitons R is simultaneously the Ricci curvature of space-time. Therefore it allows to interpret the gravitational forces as effects of a non-flat space-time geometry. But the theory has a very important defect: it is consistent only classically.

On the contrary, the Standard Model is a *bona fide* quantum theory. It is of Yang–Mills [9] type based on local non-abelian gauge symmetry group $SU(3) \times SU(2) \times U(1)_Y$ where the group $SU(3)$ is responsible for strong interactions while electroweak forces are related to $SU(2) \times U(1)_Y$. The particle spectrum contains spin 1/2 fermions in different representations and spin 1 gauge bosons constituting (8, 1), (1, 3) and (1, 1) representations of the gauge group. The gauge bosons which are directly responsible for a mediation of the interactions are described as coefficients of a connection of the gauge symmetry group. All the fermions are divided into three generations characterized by different mass scale but filling up the identical pattern of representations: $(3, 2)_{1/6}$ for left-handed quarks, $(1, 2)_{-1/2}$ for left-handed leptons, $(3, 1)_{2/3}$ and $(3, 1)_{-1/3}$ for right-handed quarks and $(1, 1)_{-1}$ for right-handed leptons (plus $(1, 1)_0$ if we include right-handed neutrinos). The theory is chiral i.e. a transformation interchanging left-handed fermions with right-handed ones is not a symmetry of the theory. The chirality forbids fermions to attain a mass from the usual Dirac mass term: $\overline{\phi}_R m \phi_L + \overline{\phi}_L m \phi_R$ because such a term would not be gauge invariant. A necessary part of the model is the Higgs mechanism [10, 11] of a spontaneous breaking of the $SU(2) \times U(1)_Y$ symmetry into the electromagnetic $U(1)_{em}$. This mechanism makes fermions and three vector bosons carrying weak interactions massive but to make the mechanism working there should necessarily exist additional, still experimentally not observed fields – scalar Higgs bosons. With the exception of this point requiring confirmation, the Standard Model is a consistent quantum theory, renormalizable, anomaly free and last but not least confirmed in all existing experiments with a fantastic precision.

2.2 The need for unification

The Standard Model and General Relativity are surprisingly successful in prediction and description of all of the experimentally observed physical phenomena at the fundamental level. They have however significant theoretical disadvantages. A validity of both of the theories is limited and both of them are constructed under assumptions that seem arbitrary from a purely theoretical point of view. What is more, some aspects of one of the theories are in contradiction with the other, the most profound example being quantum character of the Standard Model and classical one of General Relativity. Therefore one cannot obtain a consistent physical theory just by joining them together. The most popular hypothesis says that the theories are only extreme limits of some yet unknown unified theory. This theory is expected to be unified not only in a sense of its completeness in a phenomenological description of the Universe, but it also should unify all the four interactions reducing them to low energy limit symptoms of a single fundamental force. The history of physics notes similar facts in the case of electricity and magnetism in nineteenth century and in the case of electromagnetism and weak interactions thirty years ago. We believe that there are no reasons to forbid occurring it again. But before we start to follow the way leading to the unified theory let us list the main problems arising in the Standard Model or the General Relativity Theory and questions which cannot be answered by these theories.

- The main disadvantage of the Standard Model is its arbitrariness. The agreement with experiments even in the simplified version with no right-handed neutrinos requires determination of 18 seemingly arbitrary and uncorrelated parameters (3 coupling constants, 6 masses of quarks, 3 masses of leptons, 3 quark mixing angles, one phase and two parameters of the Higgs potential).
- Similarly arbitrary is a choice of the gauge group and a choice of the fields multiplets appearing in the Model (constrained only by vanishing of anomalies).
- A next mysterious thing is a number of the fermionic generations. We do not know why exactly three generations are observed, whether we should expect next generations at higher energy level and why fields in different generations have different masses.
- We also do not know why all known generations are chiral and whether the chirality is a physical rule still true for the higher generations if they exist.
- There is unexplained reason for which electric charge is quantized. For the anomaly cancellation it is enough if a certain combination of charges vanish, but the condition does not state that, for example, the up quark charge is exactly 2/3 of the positron charge.
- In classical General Relativity problems are of different nature. To find any solution one has to provide the sources for the space-time curvature i.e. the energy-momentum tensor. To describe the evolution of the Universe we have to assume that the energy-momentum tensor is extremely unnatural - besides the well understood matter and radiation content it has to contain the so called cosmological constant which in comparison with any theoretical estimates is too small by tens orders of magnitude.
- The only intrinsic parameter of General Relativity is κ – the gravitational constant. It creates even bigger problems. Since it is dimensional it describes not only a relative strength (or rather weakness) of the gravity compared to other interactions but gives a specific length and energy scale called the Planck's scale:

$$l_{Planck} \simeq 10^{-35} m \simeq (10^{-19} GeV)^{-1} \simeq M_{Planck}^{-1}. \quad (1)$$

The theory with such a scale cannot be quantized with usual methods, because when an energy of an interaction exceeds the the Planck's mass the theory becomes strongly coupled and nonrenormalizable. However following the example how the Fermi theory of weak interactions

was replaced by the Weinberg-Salam model one should expect an appearance of a new quantum physics at the Planck scale and a new theory of gravity being a generalization of General Relativity.

- Possible existence of the new physics at the Planck scale automatically causes so called hierarchy problem. Calculating quantum corrections to masses of the low energy (electroweak scale) particles we find that the corrections should be very large. Barring a possibility that these masses are just fine-tuned with extreme precision there should exist a mechanism restricting the interaction between low energy and high energy particles and then preventing a mixing of the both scales.
- At the classical level solutions of General Relativity equations of motion contain singularities (like the famous Big-Bang singularity). Therefore the theory predicts existence of objects that cannot be satisfactorily described by the theory itself and these objects may be different or even disappear in the larger theory
- Although General Relativity directly connects energy and geometry of a space-time, it does not answer a question why the observed Universe is flat and has exactly three space and one time direction. And indeed it is possible to define the theory in a space-time with arbitrary dimensionality, signature and geometry.

One could idealistically think, that to define the Unified Theory of All Interactions it should be enough to write an adequate set of assumptions determining a consistent theory. On one hand the number of assumptions should be as small as possible to answer a request for "naturalness" of the theory. But on the other the assumptions should be strong enough to allow a derivation of only a single theory which obviously has to reduce to the Standard Model and General Relativity in the low energy limit. Unfortunately till today nobody has found any nontrivial and fully consistent quantum theory let alone to write down such a set of axioms. Some physicists even doubt the possibility of formulation of the Unified Theory that way (if of course it could be formulated anyhow).

But there is another way perhaps more efficient. We should extract the principles constituting the Standard Model and General Relativity and slightly generalize or relax one or few of them. If after such a redefining it is not possible to construct a theory more unified, less arbitrary and still compatible in some limit with the starting ones, then we should step back and try with some other set of principles. If it is possible then we should suppose that it was a correct step towards the Unified Theory.

Let us describe some of the steps that presumably lead towards the Unified Theory [12].

2.3 Grand Unification Theories.

Investigating by the renormalization group methods a behavior of the three coupling constants of the Standard Model one finds that their values almost converge at some large ($\sim 10^{16}$ GeV) energy scale. The natural hypothesis is that perhaps the three coupling constants are replaced at this scale by a single constant and the group product $SU(3) \otimes SU(2) \otimes U(1)_Y$ by a single simple Lie group of larger gauge symmetry. If it really happens a real unification of strong and electroweak interactions is attained in the similar way as the electromagnetic and weak interactions are unified at ~ 100 GeV. The idea of a Grand Unified Theory (GUT) [13, 14] has several attractive theoretical features for example it can explain why the $U(1)$ charges are quantized. Let us discuss the most attractive GUT – $SO(10)$ theory [15]. A fundamental representation of $SO(10)$ is **16** and under symmetry breaking $SO(10) \rightarrow SU(3) \otimes SU(2) \otimes U(1)_Y$ it decomposes as [16]:

$$\mathbf{16} \rightarrow (\mathbf{1} \otimes \mathbf{1})_2 \oplus (\overline{\mathbf{3}} \otimes \mathbf{1})_{-\frac{4}{3}} \oplus (\mathbf{3} \otimes \mathbf{2})_{\frac{1}{3}} \oplus (\mathbf{1} \otimes \overline{\mathbf{2}})_1 \oplus (\overline{\mathbf{3}} \otimes \mathbf{1})_{-\frac{2}{3}} \oplus (\mathbf{1} \otimes \mathbf{1})_0, \quad (2)$$

where the numbers in subscripts denote values of the $U(1)_Y$ electroweak hypercharge. As we see the representations appearing in the above formula precisely agree with the representations containing spinor fields in the Standard Model (both quarks and leptons) including the right-handed neutrino.

However GUTs still suffer most of the problems peculiar for the Standard Model. All these theories are full of arbitrary parameters, do not answer the questions of the number of generations or chirality and do not propose any way to incorporate gravity. Since a new energy scale (the grand unification scale) appears in the theories which is many orders of magnitude higher than the electroweak scale, the hierarchy problem is even more pronounced. Additionally there is a lot of Lie groups containing the Standard Model symmetry group as their subgroup and therefore being possible bases for construction of different GUTs and we have no clear indication how the correct one should be chosen.

2.4 Extra dimensions.

A proposal to unify gravity with other interactions was first put forward by Kaluza and Klein in 1920s [17, 18, 19]. It consists in introducing additional dimensions of a space-time [20] and interpreting the additional geometrical symmetries of higher-dimensional gravity as the gauge symmetries of the four-dimensional world. This is not a new idea. If we look how electricity and magnetism is unified in the Maxwell theory of electromagnetism we can see that the key of the unification is to replace a three dimensional space with separated time by a four dimensional space-time. So a conjecture that a similar mechanism can be used in case of gravity seems to be very plausible.

Let us start with a D dimensional theory where d dimensions are infinite (or very large) and $D-d$ dimensions are small (i.e. the inverse radius is larger than presently available energies and therefore impossible for direct detection). In other words we assume that at a low energy level a D -dimensional space-time \mathcal{M}_D splits into a product $\mathcal{M}_d \times \mathcal{K}$, where \mathcal{K} is relatively small. A natural assumption is that a length scale of the small directions is of the order of the Planck's scale.

With such a splitting G_D – the original symmetry group of the general coordinate transformations defined on \mathcal{M} – breaks into a product $G_d \times G_{D-d}$ of symmetries defined on \mathcal{M}_d and \mathcal{K} respectively. Any field in an arbitrary representation of G_D decomposes into a sum of products of representations of $G_d \times G_{D-d}$. But for a d -dimensional observer who cannot excite momentum along \mathcal{K} , group G_{D-d} is seen as a gauge symmetry group and with different choices of topology of \mathcal{K} we can in principle obtain any gauge symmetry group.

From the d -dimensional point of view each of the resulting fields has an additional dependence on $D-d$ continuous parameters – coordinates on \mathcal{K} . If \mathcal{K} is a compact manifold it leads to very interesting conclusions. As an example let us consider a field $f(X_D)$ satisfying on \mathcal{M} an equation $\mathcal{D}_D f = 0$, where \mathcal{D}_D is an adequate wave operator. For wave operators on products of manifolds it is possible to write $\mathcal{D}_D = \mathcal{D}_d + \mathcal{D}_{D-d}$ where \mathcal{D}_d commutes with \mathcal{D}_{D-d} . Therefore:

$$f(x_D) = \sum_g f^{(g)}(x_d) v^{(g)}(y_{D-d}), \quad (3)$$

where g runs over some countable set. The $f^{(g)}$, $v^{(g)}$ are eigenvectors of the operators \mathcal{D}_d and \mathcal{D}_{D-d} respectively and constitute a complete, orthogonal systems. Then:

$$\mathcal{D}_D f(X_D) = \sum_g v^g(y_{D-d}) \left(\mathcal{D}_d + m_{D-d}^{(g)} \right) f^{(g)}(x_d), \quad (4)$$

where $m_{D-d}^{(g)}$ are eigenvalues of \mathcal{D}_{D-d} . So, in d dimensions we see a tower of the so called Kaluza–Klein modes $f^{(g)}(x_d)$ each characterized by a mass (for fermions) or mass squared (for bosons) $m_{D-d}^{(g)}$. It is plausible to identify stages of the tower with the subsequent generations of the fields.

It would be extremely appealing if one could justify the Standard Model gauge group in such a purely geometrical way – it turns out however that it is impossible with the main obstacle being the chirality of fermions in the Standard Model.

2.5 Supersymmetry and supergravity.

Looking at the Standard Model one can notice a strange asymmetry between bosons and fermions. Each kind of fields appears in the Model in different representations of the gauge group and because of

that they play different roles. Spin 1 bosons are in the adjoint representation of the gauge group and can be interpreted as carriers of interactions, while spin 1/2 fermions are in fundamental (or trivial) representations and act only as pure matter and there is no the slightest hint that bosons and fermions could be connected.

In the middle of 1970s however it was realized that in string theory the seemingly fundamental difference between bosons and fermions is alleviated and there exists there a symmetry connecting bosons and fermions – it was called supersymmetry [21, 22, 23, 24, 25]. An immense theoretical effort over the last 20 years was devoted to the application of this idea for different theories and most of the nearest future experimental efforts in particle physics is aimed at one goal – a discovery of the supersymmetric partners of the Standard Model particles.

The starting point for a construction of supersymmetry group is to assume that generators of the group are fermions, so a supersymmetry transformation of a bosonic field is gives a fermion field and vice versa. In a consequence, any supersymmetry representation contains equal number of bosonic and fermionic degrees of freedom and all fields belonging to a single representation have to have the same mass. The last statement stays in contradiction with experiments because supersymmetric partners of known particles are not observed. Therefore, if supersymmetry is realized in nature it cannot be exact but must be (spontaneously) broken. But even broken supersymmetry has many interesting features [26, 27].

The assumption of supersymmetry puts rigorous constraints on a theory highly determining a number and a kind of terms which can appear in its lagrangian. For example the cosmological term is forbidden so the cosmological constant problem is less severe (by 60 orders of magnitude) bringing down its scale to the supersymmetry breaking scale.

Thanks to the fermion-boson symmetry supersymmetric theories are usually less divergent than their nonsupersymmetric counterparts and can help to solve the hierarchy problem. The idea comes from an observation that in the perturbation expansion corrections from bosonic excitations have to be accompanied by fermionic ones and the latter contribute with an opposite sign. As a result many of the corrections cancel each other reducing the degree of divergence (for scalars from the quadratic to the logarithmic).

Supersymmetry constitutes also a doorway by which gravity can be introduced to join the other interactions. A first observation is that the supersymmetry generators belong to an algebra that necessarily contains the Poincaré algebra as its subalgebra. So we can conjecture that there should exist a supersymmetric theory being an extension of the general relativity. Indeed, if we want to construct a local supersymmetry we need a spin 3/2 vector-spinor field. They play a role of a connection of the supersymmetry group analogously like the gauge vectors are connections of the gauge symmetries groups. But the vector-spinors have to be in one supermultiplet with spin 2 fields, which naturally can be interpreted as fields carrying gravitational interactions. This is the reason why the locally supersymmetric theories are usually called supergravities [29, 30].

Supersymmetry has also another feature especially interesting in an association with extra dimensional theories [28, 31]. Denoting by N a number of irreducible spinorial supersymmetry generators one can talk about N supersymmetries. From the four-dimensional point of view acting with a supersymmetry generator changes a spin projection of the field by 1/2. It is natural not to introduce fields with spin higher than 2, because there are not known consistent interacting quantum theories describing them so the maximal number of four dimensional supersymmetries is $N = 8$. This leads to the conclusion that the highest number of dimensions with supersymmetry is generally twelve and with one time direction it is eleven (since $N = 1$ in $D = 11$ corresponds to $N = 8$ in $D = 4$).

In this way we come to the eleven dimensional supergravity [32]. The theory is constructed only from one supermultiplet $(g_{MN}, \Psi_M, C_{MNR})$ where the Ψ_M is a Majorana spinor and its action is:

$$\begin{aligned} S_{D=11} &= \int_{\mathcal{M}} d^{11}X \left\{ \sqrt{|g|} \left[R + \frac{1}{2} \bar{\Psi}_M \Gamma^{MNR} D_N \left(\frac{\omega + \hat{\omega}}{2} \right) \Psi_R - \frac{1}{48} H^{MNR} H_{MNR} \right. \right. \\ &\quad \left. \left. - \frac{1}{384} \left(\bar{\Psi}_M \Gamma^{MNRSTU} \Psi_N + 12 \bar{\Psi}^R \Gamma^{ST} \Psi^U \right) \left(H_{RSTU} + \hat{H}_{RSTU} \right) \right] \right\} \end{aligned}$$

$$+ \frac{1}{144^2} \epsilon^{M_1 \dots M_{11}} H_{M_1 \dots M_4} H_{M_5 \dots M_8} C_{M_9 \dots M_{11}} \Big\}, \quad (5)$$

where $H = dC$ is a strength of the potential C , D_M is a covariant derivative, ω a spin connection and:

$$\hat{H}_{MNRS} = H_{MNRS} + \frac{3}{2} \bar{\Psi}_{[M} \Gamma_{NR} \Psi_{S]}, \quad (6)$$

$$\hat{\omega}_{M\bar{N}\bar{R}} = \omega_{M\bar{N}\bar{R}} + \frac{1}{16} \bar{\Psi}^S \Gamma_{M\bar{N}\bar{R}ST} \Psi^T. \quad (7)$$

The supergravity transformations laws are given by:

$$\delta_\eta \Psi_M = \left[D_M(\hat{\omega}) - \frac{1}{288} (\Gamma^{NRST}{}_M - 8g_M^N \Gamma^{RST}) \hat{H}_{NRST} \right] \eta, \quad (8)$$

$$\delta_\eta e_M^{\bar{N}} = -\frac{1}{4} \bar{\eta} \Gamma^{\bar{N}} \Psi_M, \quad (9)$$

$$\delta_\eta C_{MNR} = -\frac{3}{4} \bar{\eta} \Gamma_{[MN} \Psi_{R]}, \quad (10)$$

where $e_M^{\bar{N}}$ is an elfbein and η is a spinorial parameter of supergravity. In the above we replaced the coefficient $1/2\kappa$ which should accompany the curvature term R with the number 1. The operation is allowed if we assume to work with such units system where $\kappa = 1/2$.

It is instructive to check the number of physical degrees of freedom of the fields. In arbitrary dimension they are given by (the number for a Dirac spinor should be multiplied by 2 and for a Majorana-Weyl spinor divided by 2) :

- $\#g_{MN} = \frac{1}{2}(D-2)(D-1)-1$,
- $\#C_{M_1 \dots M_n} = \frac{1}{n!}(D-2) \dots (D-n-1)$,
- $\#\Psi_M = (D-3)2^{[D/2]-1}$.

So, if $D = 11$, a simple calculation gives 44 degrees for the metric tensor and 84 for the antisymmetric rank 3 potential. Their sum is 128 and is precisely equal to the number of on-shell degrees of freedom of the gravitino.

The eleven dimensional supergravity was thought in the past as an excellent candidate for the Unified Theory. Firstly, it is simple and elegant. Secondly, quite natural, because it is naturally distinguished in a set of all supersymmetry theories as its maximal element. Moreover a compactification from eleven to four dimensions on $\mathcal{K} = \text{CP}(2) \times S^2 \times S^1$ could even produce the gauge group of the Standard Model, but the mechanism cannot give a chiral four dimensional theory because of lack of gauge symmetry in the $D = 11$ supergravity.

Let us describe also some lower dimensional supergravities. In ten dimensions there are three possible theories described as: $(N_L, N_R) = (1, 1), (2, 0), (1, 0)$ where N_L and N_R count numbers of left-handed and right-handed generators respectively.

The $(1, 1)$ theory [33, 34, 35] can be derived by a dimensional reduction form the eleven dimensional one, since the $D = 11$ Majorana spinor splits into two Majorana-Weyl spinors of opposite chirality in $D = 10$. An action of the theory truncated to only bosonic part reads:

$$S_{(1,1)} = S_{NS} + S_R + S_{CS}, \quad (11)$$

$$S_{NS} = \int_{\mathcal{M}} d^{10}X \sqrt{|g|} \left(R - \frac{1}{2} \partial_M \phi \partial^M \phi - \frac{1}{2} e^{-\phi} |H_{[3]}|^2 \right), \quad (12)$$

$$S_R = -\frac{1}{2} \int_{\mathcal{M}} d^{10}X \sqrt{|g|} \left(e^{3\phi/2} |F_{[2]}|^2 + e^{\phi/2} |\hat{F}_{[4]}|^2 \right), \quad (13)$$

$$S_{CS} = -\frac{1}{2} \int_{\mathcal{M}} F_{[4]} \wedge F_{[4]} \wedge C_{[2]}, \quad (14)$$

where ϕ is a scalar field and:

$$F_{[k]} = dA_{[k-1]} \quad \text{for } k = 2, 4, \quad (15)$$

$$H_{[3]} = dC_{[2]}, \quad (16)$$

$$\hat{F}_{[4]} = F_{[4]} + A_{[1]} \wedge H_{[3]}. \quad (17)$$

The terms S_{NS} and S_R describe respectively so called Neveu-Schwarz-Neveu-Schwarz and Ramond-Ramond sectors of the theory (the meaning of this terminology will be explained later) and S_{CS} is Chern-Simons term.

The $(2, 0)$ supergravity [36, 37] is at first sight quite odd. It cannot be derived from dimensional reduction of eleven dimensional supergravity. Moreover it is impossible to write down a lagrangian because the theory contains an antisymmetric field of rank 5, which is selfdual i.e.:

$$\hat{F}_{[5]} = * \hat{F}_{[5]}. \quad (18)$$

However, the equations of motion derived from an action with bosonic part given by:

$$S_{(2,0)} = S_{NS} + S_R + S_{CS}, \quad (19)$$

$$S_{NS} = \int_{\mathcal{M}} d^{10}X \sqrt{|g|} \left(R - \frac{1}{2} \partial_M \phi \partial^M \phi - \frac{1}{2} e^{-\phi} |H_{[3]}|^2 \right), \quad (20)$$

$$S_R = -\frac{1}{2} \int_{\mathcal{M}} d^{10}X \sqrt{|g|} \left(e^{2\phi} |F_{[1]}|^2 + e^\phi |\hat{F}_{[3]}|^2 + \frac{1}{2} |\hat{F}_{[5]}|^2 \right), \quad (21)$$

$$S_{CS} = -\frac{1}{2} \int_{\mathcal{M}} A_{[4]} \wedge H_{[3]} \wedge F_{[3]}, \quad (22)$$

where:

$$F_{[k]} = dA_{[k-1]} \quad \text{for } k = 1, 3, 5, \quad (23)$$

$$H_{[3]} = dC_{[2]}, \quad (24)$$

$$\hat{F}_{[3]} = F_{[3]} - A_{[0]} \wedge H_{[3]}, \quad (25)$$

$$\hat{F}_{[5]} = F_{[5]} - \frac{1}{2} A_{[2]} \wedge H_{[3]} + \frac{1}{2} C_{[2]} \wedge F_{[3]} \quad (26)$$

with (18) as an extra condition are just the equations of motion of the $(2, 0)$ supergravity.

Setting equal to zero one of the generators of the $(1, 1)$ or $(2, 0)$ theories one reduces them to the $(1, 0)$ theory [38, 39]. The bosonic part of its action is:

$$S_{(1,0)} = \int_{\mathcal{M}} d^{10}X \sqrt{|g|} \left(R - \frac{1}{2} \partial_M \phi \partial^M \phi - \frac{1}{2} e^{-\phi} |F_{[3]}|^2 \right). \quad (27)$$

There one more supergravity known in ten dimensions being an extension of $(1, 1)$ theory, called massive supergravity or Romans [40] theory. An action of the theory can be obtained from the action $S_{(1,1)}$ given in (11) by adding a scalar field M and a 10-form $F_{[10]}$ in the following way:

$$S_{(1,1)massive} = \tilde{S}_{(1,1)} - \int_{\mathcal{M}} \frac{1}{2} d^{10}X \sqrt{|g|} e^{5/2\phi} M^2 + \int_{\mathcal{M}} M F_{[10]}, \quad (28)$$

where $\tilde{S}_{(1,1)}$ is the same as (11) after the substitution:

$$F_{[2]} \rightarrow F_{[2]} + MC_{[2]}, \quad (29)$$

$$F_{[4]} \rightarrow F_{[4]} + \frac{1}{2} MC_{[2]} \wedge C_{[2]}. \quad (30)$$

Field M is as an auxiliary one and can be integrated over in the action giving quite complicated combination of other fields.

As we see in 11 dimensions a supersymmetric theory is necessarily a supergravity but in 10 dimensions a supersymmetry generator can be a Majorana–Weyl spinor leading to a gauge supermultiplet (with fields of spin not higher than 1). There exists a rule that chiral Yang–Mills and simultaneously supersymmetric theories are allowed only if $N = 1$. In such a case nothing prevents the $(1, 0)$ supergravity for coupling to matter which is supersymmetric, gauge symmetric but chirally asymmetric [39]. Such theories can be compactified to a chiral, gauge symmetric lower dimensional theory, if the supersymmetry preserved in the target space is not higher than $N = 1$. In the case of the $(1, 0)$ supergravity the required compactification scheme is obtained if \mathcal{K} is Calabi–Yau manifold (i.e. a complex manifold of complex dimension 3 and with $SU(3)$ holonomy group [41]). The reasoning is as follows: we have to compactify 6 dimensions; the holonomy group in 6 dimensions is $SO(6)$ which is locally isomorphic to $SU(4)$. If the holonomy of the manifold is $SU(3)$ (i.e. it is a Calabi–Yau manifold) then the holonomy breaks three out of four supersymmetries which can emerge after compactification from 10 to 4 dimensions. Simultaneously we can identify a subgroup of the gauge group with the holonomy group so it gives a mechanism of gauge symmetry breaking. For example starting with $E(8)$ gauge symmetry, which is a natural candidate because of reasons described later, one obtains:

$$E(8) \rightarrow E(6) \otimes SU(3) \rightarrow E(6) \quad (31)$$

and afterwards:

$$E(6) \rightarrow SO(10) \otimes U(1). \quad (32)$$

But $SO(10)$ is one of the most suitable GUT theory thanks to the behavior of its multiplets under reduction to $SU(3) \otimes SU(2) \otimes U(1)$, (see 2).

In spite of their attractive features supergravities have a major drawback – they are nonrenormalizable. They are consistent only at a classical level but not at the quantum one. But even if they are not the final goal one is tempted to think that they indicate the correct way toward the Unified Theory.

2.6 Bosonic strings.

The result of search for the unification of all interactions summarized in the previous section can seem quite disappointing. Even if we recovered an interesting idea that could lead to unification of gravity with other fundamental forces it turned out that a theory incorporating the idea was still nonrenormalizable. But for now we were considering only theories where the fundamental objects were defined as point particles. What if the point-like structure is only a simplification, the particles can be extended objects and at some scale (for example the Planck’s scale) it is necessary to take into account this fact? So, let us describe a theory with points replaced by strings [42, 43, 44]. Then the Feynman diagrams, which in the case of point particles were networks of crossing worldlines now become homomorphic to two dimensional manifolds. But at any order of the perturbation expansion a number of topologically inequivalent two-folds is significantly smaller than a number of topologically inequivalent line-like diagrams, so the theories of strings at first sight seem to be more convergent than the point particles’ ones. And really they are. They are only known examples of consistent interacting quantum theories which include both gauge interactions and gravity, they are expected to be finite at every level of perturbation expansion and they are not anomalous. But this is not the only advantage of the string theories. The other include:

- They contain only two dimensionful parameters – speed of light c and the string scale λ_s .
- They can be constructed only in a specific dimension of a spacetime.
- Fields of various spins and masses are quantum excitations of a single string.
- At the low energy limit string theories reduce to supergravities possibly coupled to super Yang Mills.

- There is only a small number of inequivalent string theories and connections among them (dualities) suggest that they are all only special cases of one theory (M-theory).

Let us describe in more detail a construction of the string theories. It is natural to start with an analog of a relativistic point particle action $S = -m \int ds$ (so called Nambu-Goto action [45, 46]):

$$S_{NG} = -\frac{1}{2\pi\alpha'} \int d^2\xi \sqrt{|\det(\partial_a X^M \partial_b X^N \eta_{MN})|}, \quad (33)$$

where $X^M(\xi^a)$ are space-time coordinates giving a localization of the string worldsheet with respect to two parameters: timelike ξ^1 and spacelike ξ^2 , indices a, b run over values 1 and 2. The α' is a string coupling constant and it is related to a string tension by $T = 1/(2\pi\alpha') = 1/\lambda_s^2$. Because equations of motion derived from the action (33) could be equivalently obtained from another action which is free of roots of X^μ , it is more convenient to construct string theories on a base of so called Polyakov action [47]:

$$S_P = -\frac{1}{4\pi\alpha'} \int d\xi^2 \sqrt{|\gamma|} \gamma^{ab} \partial_a X^M \partial_b X^N \eta_{MN}. \quad (34)$$

In the above we introduced a worldsheet metric $\gamma_{ab}(\xi)$. The action is invariant under general transformations of coordinates ξ and global Poincaré transformations in space-time. Additionally it exhibits local Weyl symmetry given by:

$$X'^M(\xi) = X^M(\xi), \quad (35)$$

$$\gamma'_{ab}(\xi) = e^{2\omega(\xi)} \gamma_{ab}(\xi). \quad (36)$$

This is an important fact, because three parameters of the local symmetries exactly agree with a number of γ_{ab} degrees of freedom. Therefore the field describing internal structure of a string can be completely gauged out from the physical theory and even in the quantum theory we have well defined distances on the world-sheet. Thanks to (35 - 36) a worldsheet of a propagating string can be described not only as a general two dimensional real manifold but also as a Riemann surface, it means as a complex one-fold. This is a starting point for developing so called conformal field theory (CFT) which gives an apparatus allowing to evaluate an amplitude for any scattering process in the string theory. In such a description there are deep subtleties in the Wick rotation from the physical Minkowski to the Euclidean signature on the world sheet - it is a beautiful mathematical result that such a rotation can be done and that the amplitudes coincide. This result justifies the common approach to string theory by the powerful formalism of conformal field theory.

One distinguishes several categories of strings characterized as open or closed and oriented or unoriented. The open ones are homeomorphic to an open interval and have to satisfy the following boundary conditions:

$$\frac{\partial}{\partial\xi^2} X^M(\xi^1, 0) = 0 = \frac{\partial}{\partial\xi^2} X^M(\xi^1, l), \quad (37)$$

where $\xi^2 = 0$ and $\xi^2 = l$ describe free ends of the string. Analogously the closed ones are homeomorphic to a circle, so the points $\xi^2 = 0$ and $\xi^2 = l$ should be identified in this case and the boundary conditions are:

$$X^M(\xi^1, \xi^2) = X^M(\xi^1, \xi^2 + l), \quad (38)$$

$$\frac{\partial}{\partial\xi^2} X^M(\xi^1, \xi^2) = \frac{\partial}{\partial\xi^2} X^M(\xi^1, \xi^2 + l). \quad (39)$$

The open string conditions are more restrictive. While on the closed strings there can live two infinite series of quantum excitations related to left and right moving waves

$$X^M(\xi^1, \xi^2) = X_L^M(\xi^2 - \xi^1) + X_R^\mu(\xi^2 + \xi^1) \quad (40)$$

where X_L^M and X_R^M are independent, on the open strings they are combined into one series of stable waves.

The oriented and the unoriented strings are defined as strings with worldsheets of respectively oriented or unoriented manifolds. The orientability is a global feature of manifolds and in the interacting theory all the string worldsheets are joined and form a single connected manifold. So, it leads to a conclusion that the oriented and the unoriented strings can interact only within their classes. By use of a purely topological arguments it can be checked that an interaction among open strings always can produce a closed one, while for the closed strings it is possible to impose consistent restrictions forbidding them to interact with the open ones. Therefore there are four possible kinds of interacting string theories:

- theories with only closed oriented strings,
- theories with only closed unoriented strings,
- theories with open and closed strings, all oriented,
- theories with open and closed strings, all unoriented.

Quantizing the theory derived from (34) with an assumption that it should preserve Lorentz invariance one obtains a spectrum of quantum states with masses given by:

$$m^2 = \frac{s^2}{\alpha'} \left(N + \frac{2 - D}{24} \right), \quad (41)$$

where D is a dimension of space-time, N enumerates levels of excitations, and s is a number equal to 1 in the case of open strings and 2 in the case of closed strings. Counting number of states at each level and checking their behavior under Lorentz transformations one finds that only $N = 1$ can form massless representations in D dimensional space-time. Simultaneously, from the condition $m^2 = 0$ for $N = 1$ one obtains $D = 26$ as a critical dimension i.e. the number of dimensions where strings can live. For the oriented open strings the massless representation is a vector and for the oriented closed strings it is a multiplication of two vectors which decomposes into irreducible rank 2 tensors: a traceless symmetric tensor, an antisymmetric tensor and a scalar. The condition of unorientability reduces a number of possible massless fields disallowing vectors and antisymmetric tensors.

Consider an interaction among several external open strings. Because ends of the external strings are distinguished points on a worldsheet, each possible interaction can be characterized by a specific order of the points and this order is invariant under the worldsheet reparametrization. This means that the open strings in a distinction to the closed ones have additional degrees of freedom attached to their endpoints. They are known as Chan-Patton degrees of freedom and can be described in terms of gauge symmetry [48]. More detailed analysis shows that possible gauge groups are $U(N)$ in the case of oriented string and $SO(N)$ or $Sp(N)$ in the case of unoriented strings.

Until now we have discussed strings in flat space-time but it is possible to consider strings propagating in other backgrounds. A first step of such an extension is to replace the constant flat metric η_{MN} from (34) by a general coordinate dependent metric $g_{MN}(X)$ and add couplings to other massless states of the oriented closed string: the antisymmetric tensor $C_{MN}(X)$ and scalar $\phi(X)$. Then one obtains a nonlinear sigma model action which takes the form:

$$S_P = -\frac{1}{4\pi\alpha'} \int d^2\xi \sqrt{|\gamma|} [(\gamma^{ab} g_{MN}(X) + i\epsilon^{ab} C_{MN}(X)) \partial_a X^M \partial_b X^N + \alpha' R[\gamma] \phi(X)]. \quad (42)$$

The above theory is a renormalizable theory of fields $X^M(\xi)$. It is consistent only if Weyl invariance is preserved on a quantum level, it means when the following conditions leading to a cancellation of Weyl anomaly are satisfied:

$$0 = \alpha' \left(R[g]_{MN} + 2\nabla_M \nabla_N \phi - \frac{1}{4} H_{MNR} H^{MNR} \right) + O(\alpha'^2), \quad (43)$$

$$0 = \alpha' \left(-\frac{1}{2} \nabla^M H_{MNR} + \nabla_M \phi H^{MNR} \right) + O(\alpha'^2), \quad (44)$$

$$0 = \alpha' \left(-\frac{1}{2} \nabla^2 \phi + \nabla^M \phi \nabla_M \phi - \frac{1}{24} H_{MNR} H^{MNR} \right) + O(\alpha'^2), \quad (45)$$

where $H = dC$. These conditions look like equations of motion of some theory and indeed it is possible to find an action from which they can be derived:

$$S_{string} = \int d^{26}x \sqrt{|g|} e^{-2\phi} \left[R[g] - \frac{1}{12} H_{MNR} H^{MNR} + 4\partial_M \phi \partial^M \phi + O(\alpha') \right]. \quad (46)$$

In this way one arrives at an effective theory describing massless modes of the closed oriented string. The above action is written in a formalism known as a string frame, in which the curvature term is given by $\sqrt{|g|} e^{-2\phi} R[g]$. By a redefinition of fields:

$$g_{MN} \rightarrow \exp \left(-\frac{\tilde{\phi}}{\sqrt{2(D-2)}} \right) \tilde{g}_{MN}, \quad (47)$$

$$\phi \rightarrow \sqrt{\frac{D-2}{8}} \tilde{\phi} \quad (48)$$

it can be transformed to so called Einstein frame where the curvature term is $\sqrt{|\tilde{g}|} R[\tilde{g}]$ and the whole action:

$$S_{Einstein} = \int d^{26}x \sqrt{|\tilde{g}|} \left[R[\tilde{g}] - \frac{1}{12} e^{-\tilde{\phi}/\sqrt{3}} \tilde{H}_{MNR} \tilde{H}^{MNR} + \frac{1}{2} \partial_M \tilde{\phi} \partial^M \tilde{\phi} + O(\alpha') \right]. \quad (49)$$

However the theory (49) is obviously not a good candidate for an effective string theory being simultaneously a generalization of the Standard Model and General Relativity, because it does not contain fermions. Another important disadvantage is that the fields at the lowest level in the spectrum given by (41) are not massless but tachyonic with a negative mass square described by $N = 0$. So, the interacting theory cannot be stable since any excited state including the massless ones should decay into tachyons. It was conjectured that the presence of the tachyons is a consequence of a wrong vacuum choice and there should be a mechanism shifting the theory to the correct vacuum similarly as the Higgs mechanism makes it with the Standard Model. But this idea did not give a satisfactory result.

2.7 Superstrings and M-theory.

Fortunately there is another method avoiding the shortcomings of the bosonic string theory described above but saving its virtues. The key idea is to apply supersymmetry to strings and therefore introduce so called superstrings. There are two variants of the theory, each more convenient in different but complementary aspects. Fortunately both lead to at least partially equivalent results.

The first one is known as spacetime supersymmetry or Green-Schwarz theory [49, 50, 51]. In this case derivatives $\partial_a X^M$ in (34) are replaced by:

$$\Pi_a^M = \partial_a X^M - i\bar{\theta}^\alpha \gamma^M \partial_a \theta^\alpha \quad (50)$$

and the resulting action is:

$$S_{GS} = -\frac{1}{4\pi\alpha'} \int d\xi^2 \sqrt{|\gamma|} \gamma^{ab} \Pi_a^M \Pi_b^N \eta_{MN}. \quad (51)$$

where θ is a spinor in D dimensions, $\alpha = 1, \dots, N$ with N giving a number of global supersymmetries. The expression (50) is invariant under transformations of the supersymmetries:

$$\delta_\epsilon \theta^\alpha = \epsilon^\alpha, \quad \delta_\epsilon \bar{\theta}^\alpha = \bar{\epsilon}^\alpha, \quad \delta_\epsilon X^M = i\bar{\epsilon}^\alpha \Gamma^M \theta^\alpha. \quad (52)$$

However it can be checked, that the θ^α fields have twice too many components than can be determined by solving the equations of motion of the theory. So, the additional symmetry is needed to gauge away the undesired degrees of freedom and possibly redefining (51) to incorporate new terms allowing a closure of the action under the new symmetry.

To write the appropriate action it is convenient to introduce a superspace formalism with supercoordinates: $Z^M = (X^M, \theta^\alpha)$ and a supervelbein $E_{\bar{M}}^{\bar{M}}$ where the indices $\bar{M} = (\bar{M}, \alpha)$ label tangent space coordinates. Define $E_a^{\bar{M}} = \partial_a Z^M E_M^{\bar{M}}$ and then the action reads:

$$S_{GS} = \frac{1}{4\pi\alpha'} \int d^d\xi \left(-\frac{1}{2} \sqrt{|\gamma|} \gamma^{ab} E_a^{\bar{M}} E_b^{\bar{N}} \eta_{\bar{M}\bar{N}} + \frac{1}{d!} \epsilon^{ab} E_a^{\bar{M}} E_b^{\bar{N}} B_{\bar{M}\bar{N}} \right). \quad (53)$$

The action is invariant under local fermionic transformations called κ symmetry:

$$\delta_\kappa Z^M E_M^{\bar{M}} = 0 \quad \delta_\kappa Z^M E_M^\alpha = (1 + \Gamma)_\beta^\alpha \kappa^\beta(\xi), \quad (54)$$

where

$$\Gamma_\beta^\alpha = \frac{-i}{2! \sqrt{|\gamma|}} \epsilon^{ab} E_a^{\bar{M}} E_b^{\bar{N}} (\Gamma_{\bar{M}\bar{N}})_\beta^\alpha. \quad (55)$$

It is important that the action (53) has to incorporate the Wess-Zumino term with antisymmetric tensor to achieve κ invariance. But on the other hand the action is not supersymmetric for arbitrary N and D as (51) is. It can be checked that such a situation occurs only for $N \leq 2$ and $D = 3, 4, 6, 10$.

A quantization program for the Green-Schwarz theory encounters serious difficulties and it was carried out only in the light-cone gauge. But there is another supersymmetrisation method for strings, more convenient for the quantization but with the manifest space-time supersymmetry lost. The method is known as world-sheet supersymmetry or Ramond-Neveu-Schwarz theory [52, 53, 54, 55]. The idea in this case is to add to (34) fields ψ^M being Majorana spinors on the worldsheet but vectors in the spacetime:

$$S_{world\ sheet\ SUSY} = -\frac{1}{4\pi\alpha'} \int d\xi^2 \sqrt{|\gamma|} \gamma^{ab} \left(\partial_a X^M \partial_b X^N - i \bar{\psi}^M \gamma_a \partial_b \psi^N \right) \eta_{MN}. \quad (56)$$

Then, for any given M the pairs (X^M, ψ^M) are scalar supermultiplets of $N = 1$ supersymmetry. Quantizing the theory one obtains a critical dimension $D = 10$. It is also possible to consider extended $N > 1$ supersymmetries. But it occurs that for $N = 2$ the critical dimension is $D = 2$ what is unreasonable for a theory with eventual applications but can be interesting as a playground for theoretical experiments. For $N > 2$ the critical dimension is negative what is obviously unacceptable.

The spinors living on a string worldsheet have to obey one of two possible boundary conditions, known as Ramond (R) and Neveu-Schwarz (NS) ones:

$$\begin{aligned} \psi(\xi^1 + l, \xi^2) &= +\psi(\xi^1, \xi^2) && R, \\ \psi(\xi^1 + l, \xi^2) &= -\psi(\xi^1, \xi^2) && NS, \end{aligned} \quad (57)$$

what gives two kinds of quantum states. Additionally all quantum states can be divided with respect to the worldsheet fermion number operator $(-1)^F$ being an extension of the chirality operator with two eigenvalues $+1$ and -1 . The projection from the space of all states onto states of given fermion number is called the GSO (Gliozzi-Scherk-Olive) projection.

Identically as for the bosonic strings, on the open superstrings only stable waves can exist, so the whole quantum spectrum can be classified by four sectors labelled as R+, R-, NS+ and NS-. The lowest state in NS- is a tachyon, but in NS+, R-, R+ it is a massless vector and two massless Majorana-Weyl spinors with opposite chirality respectively. On the closed superstrings left and right moving waves are independent of each other, so the spectrum in this case is given by sectors described as pairs of the open superstrings sectors. Total number of the closed superstring sectors is 10 and not 16 because combinations of NS(-) with the other possibilities are forbidden by the quantum level matching rule. The rule says that tachyons are only in (NS-,NS-) sector and the lowest states in the

remaining sectors are massless. Massless fermionic states are contained in the sectors labelled by NS+ accompanied by R+ or R-, while the sectors with massless bosonic states are described by (NS+,NS+) and all possible parings among R+'s and R-'s. In short all massless bosons can be classified as NS-NS or R-R.

Each choice of several sectors can lead to a different superstring theory. A number of such choices is possibly very huge. For example in a case of closed superstrings it is equal to 2^{10} . Fortunately only a few choices give a consistent interacting theory with tachyon free and supersymmetric spectrum.

Type IIA and IIB superstrings.

Those are closed oriented superstrings theories with two supersymmetries. They are constructed of the following sectors:

$$IIA \quad (NS+, NS+) \quad (R+, NS+) \quad (NS+, R-) \quad (R+, R-), \quad (58)$$

$$IIB \quad (NS+, NS+) \quad (R+, NS+) \quad (NS+, R+) \quad (R+, R+). \quad (59)$$

In the above $R+$ can be replaced by $R-$, what changes chirality of all fermions, but leads to equivalent physical theories. What is important, in the type IIA theory left and right moving massless fermion states have opposite chirality but in the type IIB theory all fermions have the same chirality. In the language of the GSO projection for the IIB theory the same GSO projection is chosen for both left and right moving states, while for the IIA the opposite ones.

By the low energy limiting procedure analogous to that described for bosonic string one can derive effective theories for massless states of type IIA and IIB superstrings and find that they coincides respectively with the (1,1) and (2,0) supergravities in ten dimensions. Because massless bosonic states in the superstring theory belong to two disjoint sectors R-R or NS-NS the corresponding fields in the supergravity theory can be described in the same way. This is an explanation of the classification given in (11) and (19).

Type I $SO(32)$ superstrings.

This is a theory of unoriented closed and open strings and possesses only one supersymmetry. It can be obtained from the IIB theory by the requirement of unorientability and removing states not satisfying this condition. Particularly from the two sectors containing massless fermions only one linear combination survives: $(NS,R)+(R,NS)$. But the theory is inconsistent unless open string states NS+ and R+ are also included. The additional states have Chan-Paton degrees of freedom, so the theory is gauge symmetric with the gauge group $SO(N)$ or $Sp(N)$. The requirement of vanishing anomaly further constrains the gauge group to only $SO(32)$. In the low energy limit the theory converges into the (1,0) supersymmetry coupled with $SO(32)$ super-Yang-Mills.

Heterotic $E(8) \otimes E(8)$ and $SO(32)$ superstrings.

Another possibility to construct a superstring theory is to combine the superstring constraints on a half of the closed strings states (say: on the right-moving) and the bosonic string constraints on the second half (left-moving) in a way which leads to removing tachyons from the physical spectrum. Because superstring lives in ten dimensions and the bosonic string in 26, the additional 16 dimensions of bosonic string should be compactified. This process leads to a gauge symmetry. Only two choices of the gauge group are acceptable to achieve a consistent theory with tachyon free and supersymmetric spectrum: $E(8) \otimes E(8)$ and $SO(32)$. In the low energy limit these theories become the (1,0) supersymmetry coupled with $E(8) \otimes E(8)$ or $SO(32)$ super-Yang-Mills respectively. The massless spectrum of type I and heterotic $SO(32)$ theories coincide, however they are different if the massive states are taken into account.

In this way one finally obtains exactly five consistent string theories which satisfy conditions requiring supersymmetric and tachyon free spectrum. If the conditions are relaxed, other string theories can be introduced as type 0 theory or a few heterotic theories with gauge groups other than the described before. In most cases they are simultaneously not supersymmetric and tachyonic, but there is one exception: the heterotic $SO(16) \otimes SO(16)$ superstring which is not supersymmetric but still tachyon-free.

An extensive research on string theories has recently shown, that these five consistent string theories are all connected by a net of dualities. There are two kinds of such dualities. One is T-duality [56], which is perturbative, what means that it works precisely at every level of perturbative expansion. T-duality connects theories compactified on n-torus and is described in general by $O(n, n, \mathbf{Z})$ group. In the simplest case it connects a theory compactified on a circle of radius R with another theory compactified on a circle of radius $R' = \alpha'/R$. The second kind of duality: S-duality [57, 58] is nonperturbative and it is described by $SL(2, \mathbf{Z})$ group. In particular it establishes relations between weakly and strongly coupled limits of two theories. Let us list examples of the dualities:

- Heterotic $E(8) \otimes E(8)$ theory is T-dual to heterotic $SO(32)$ theory.
- Type IIA and type IIB theory compactified on odd dimensional tori are T-dual.
- Type IIA and type IIB theories compactified on even dimensional torus are T-selfdual.
- Type I $SO(32)$ theory compactified on odd dimensional torus is T-dual to type II A theory.
- Type I $SO(32)$ theory compactified on even dimensional torus is T-dual to type II B theory.
- Type IIB theory is S-selfdual,
- Type I $SO(32)$ and heterotic $SO(32)$ theories are S-dual.

In the above no S-dual partners for type IIA and heterotic $E(8) \otimes E(8)$ superstrings are shown since these cases need a special treatment. Applying S-duality to these theories one finds that their duals do not coincide with any known superstring theory. Moreover the dual theories seem to live in 11 rather than 10 dimensions. Thanks to duality between type IIA and heterotic $E(8) \otimes E(8)$ superstrings one deduces that their S-duals have to be the same theory for which the name was coined: M-theory. However, besides a name the theory is generally unknown. We can describe only several features of it based on the S-duality relation with superstrings and conjecture that its low energy limit should be the only allowed 11 dimensional supergravity (5). So we can add to the previous list:

- Type IIA theory is S-dual to M-theory compactified on a circle S^1 .
- Heterotic $E(8) \otimes E(8)$ theory is S-dual to M-theory compactified on an orbifold S^1/\mathbf{Z}_2 .

Existence of the net of dualities among all known superstring theories leads naturally to the conclusion that all these theories are only specific sectors of some really unified, probably unique theory. The theory obviously should be at least 11 dimensional and is usually called M-theory [59, 60, 62, 61] identically as the previously introduced dual partner for type IIA and heterotic $E(8) \otimes E(8)$ superstrings. In this picture S and T dualities should be understood as symmetries of M-theory transforming one of its sectors into another. There is a conjecture that S and T dualities are only subgroups of more general symmetry group of the whole theory called U-duality.

3 Branes in quantum theories.

If it is possible to construct string theories one could wonder why do not introduce a theory of higher dimensional objects like membranes? Such an idea is not a new one. In 1962 Dirac proposed a model where the elementary particles were described in terms of modes of a vibrating membrane [63]. The membrane Dirac theory was based on a action which was a straight analog of the Nambu-Goto string action (33). Curiously, that Nambu and Goto have written their actions a few years after Dirac. But during the next year, when the string theory was rapidly developing, attempts to develop the competitive idea of membranes ended without success. The situation has changed in 1986 when a supersymmetric membrane was discovered by Hughes, Lu and Polchinski [64]. But the real breakthrough was paradoxically made on a ground of the string theory, when Polchinski [65] has shown that superstrings necessarily have branes in the spectrum as sources of Ramond-Ramond charges in the theory.

3.1 Fundamental p -branes.

Constructing the brane theory it is natural to follow the derivation of string theory. One should then start with the Nambu–Goto-like action:

$$S_{NG} = -T_d \int d^d \xi \sqrt{|\det(\partial_a X^M \partial_b X_N \eta_{MN})|} \quad (60)$$

or the Polyakov-like action [47, 66] of a p -dimensional object sweeping out in space-time a $d = p + 1$ dimensional worldvolume:

$$S_P = -\frac{T_d}{2} \int d^d \xi \sqrt{|\gamma|} (\gamma^{ab} \partial_a X^M \partial_b X^N \eta_{MN} - (d-2)). \quad (61)$$

The extended objects that after supersymmetrization and quantization would be described by such a theory are known as p -branes. The theory of superstrings should appear in this picture as a theory of 1-branes and be only a specific case of the theory. However it turns out that the superstring $p = 1$ case is specific and very difficult to generalize to arbitrary p . A first evidence of the difficulties is the worldvolume cosmological term $\sqrt{|\gamma|}(d-2)$ in (61) which vanishes for strings but survives and breaks Weyl symmetry (35 - 36) in other cases.

As for string theory there are two ways of introducing supersymmetry – by imposing worldvolume supersymmetry and further requirement of spacetime supersymmetry (a spinning brane) or by imposing spacetime supersymmetry from the very beginning (Green-Schwarz construction).

In the case of a spinning brane [66, 67] the presence of the worldvolume cosmological term precludes starting from (61) (so called no-go theorem for spinning membranes). The modified action can be introduced [68] which exhibits Weyl invariance and is classically still equivalent to (60) and (61):

$$S_W = -T_d \int d^d \xi \sqrt{|\gamma|} \left(\frac{1}{d} \gamma^{ab} \partial_a X^M \partial_b X^N \eta_{MN} \right)^{d/2}. \quad (62)$$

But even this Weyl invariant action does not lead to a fully successful theory of a spinning membrane.

The second possibility of supersymmetrisation is the Green-Schwarz spacetime supersymmetry [64, 69]. This construction is an extension of (53), so similarly as in that case there should be introduced the supercoordinates $Z^{\mathbf{M}} = (X^M, \theta^\alpha)$, the supervielbein $E_{\mathbf{M}}^{\overline{\mathbf{M}}}$ with $\overline{\mathbf{M}} = (\overline{M}, \alpha)$ labeling tangent space coordinates and $E_a^{\overline{\mathbf{M}}} = \partial_a Z^{\mathbf{M}} E_{\mathbf{M}}^{\overline{\mathbf{M}}}$. Then the action is:

$$S_{GS} = T_d \int d^d \xi \left[\frac{\sqrt{|\gamma|}}{2} \left(-\gamma^{ab} E_a^{\overline{M}} E_b^{\overline{N}} \eta_{\overline{M}\overline{N}} + (d-2) \right) + \frac{1}{d!} \epsilon^{a_1 \dots a_d} E_{a_1}^{\overline{M}_1} \dots E_{a_d}^{\overline{M}_d} C_{\overline{M}_d \dots \overline{M}_1} \right] \quad (63)$$

and it is invariant under κ symmetry:

$$\delta_\kappa Z^{\mathbf{M}} E_{\mathbf{M}}^{\overline{\mathbf{M}}} = 0, \quad \delta_\kappa Z^{\mathbf{M}} E_{\mathbf{M}}^\alpha = (1 + \Gamma)_\beta^\alpha \kappa^\beta(\xi), \quad (64)$$

where

$$\Gamma_{\beta}^{\alpha} = \frac{(-1)^{d(d-3)/4}}{d! \sqrt{|\gamma|}} \epsilon^{a_1 \dots a_d} E_{a_1}^{\overline{M}_1} \dots E_{a_d}^{\overline{M}_d} (\Gamma_{\overline{M}_1 \dots \overline{M}_d})_{\beta}^{\alpha}. \quad (65)$$

The action preserves worldvolume supersymmetry only for certain triplets of D , d and N .

When the super- p -brane is moving in a spacetime it sweeps out a d -dimensional worldvolume. It is convenient to set spacetime coordinates as:

$$X^M(\xi) = (X^a(\xi), Y^m(\xi)), \quad \text{where} \quad X^a(\xi) = \xi^a. \quad (66)$$

The super- p -brane has therefore exactly $D - d$ bosonic degrees of freedom. To count fermionic degrees of freedom we introduce n as the number of worldvolume supersymmetries and m as the number of real components of an irreducible spinor in a given worldvolume dimension. Then the number of fermionic degrees freedom on-shell is $mn/2$. But calculating the same for spacetime fermions one should take $MN/4$ (where N is a number of spacetime supersymmetries and M a number of real components of an irreducible spinor in a given spacetime dimension) because the κ symmetry halves a number of physical degrees of freedom. The condition of equality of the number of bosonic and fermionic degrees of freedom for the Green-Schwarz super- p -branes is therefore:

$$D - d = \frac{1}{2} mn = \frac{1}{4} MN. \quad (67)$$

Using the known numbers of dimensions of irreducible spinor representations with Lorentzian signature we find that the condition is satisfied by four fundamental solutions [70]:

- "octonionic" branes with $D = 11$, $d = 3$, $m = 2$, $n = 8$, $M = 32$, $N = 1$
- "quaternionic" branes with $D = 10$, $d = 6$, $m = 8$, $n = 1$, $M = 16$, $N = 1$
- "complex" branes with $D = 6$, $d = 4$, $m = 4$, $n = 1$, $M = 8$, $N = 1$
- "real" branes with $D = 4$, $d = 3$, $m = 2$, $n = 1$, $M = 4$, $N = 1$

The fundamental solutions are maximal ones in four series and can be denoted as (D_{max}, d_{max}) . Other members of the series can be obtained by a double reduction of k dimensions $(D, d) = (D_{max}-k, d_{max}-k)$ for $k = 1, \dots, d_{max}-1$. Note that for $d > 2$ all found super- p -branes have $N = 1$.

The case $d = 2$ is special and requires more detailed analysis. The reduction of the fundamental solutions gives superstrings in four possible spacetime dimensions: $D = 3, 4, 6, 10$. All of them have $N = 2$, so they are type II superstrings. But for $d = 2$ the (67) is not the only possibility. In this case it is allowed to treat left and right moving modes independently and apply supersymmetry only to one of them. Then instead of (67) another condition should be satisfied:

$$D - 2 = n = \frac{1}{2} MN \quad (68)$$

leading to $D = 3, 4, 6, 10$ solutions with $N = 1$ corresponding to heterotic superstrings.

It is worth to note, that the maximal spacetime dimension obtained in this procedure is $D = 11$ being in excellent agreement with the analogous result derived before for supergravity theories. Moreover in the super- p -branes the condition $D \leq 11$ is derived without the restriction that spin is not bigger than 2.

The conditions (67) and (68) are valid only in the case when it is assumed that the worldvolume fields form scalar multiplets. One can relax this assumption and introduce on the worldvolume other supersymmetry representations (i.e. vector or tensor supermultiplets) with additional fields. Then one can define more super- p -branes of various kinds and the table 1 shows the result of such a search [71]. However it should be noted, that matching fermionic and bosonic degrees of freedom is only a necessary but not a sufficient condition that a supersymmetric theory exists in a general case. To prove

D	11	10	9	8	7	6	5	4	3	2	1	d
	.	.	S	.	.	T	1
V	S,V	V	V	V	S,V	V	V	V	V	V	V	2
S	S	3
.	.	.	S	4
.	.	S	.	.	.	T	5
V	S,V	V	S,V	V	V	V	6
S	.	S	7
V	S,V	S,V	V	8
S,V	S,V	V	9
S	10
.	11

Table 1: The brane scan. S - scalar; V - vector; T - tensor.

that the presumable super- p -brane really exists, an analog of (63) should be written and examined in each case.

Several cases in the table seem to be especially interesting. Let us take $N=1$ super-5-brane in $D = 10$. The action (63) of the super-5-brane has to contain an antisymmetric tensor potential of rank 6 with 7-form field strength. The case is very interesting because there is a dual formulation of $(1,0)$ supergravity where the 3-form field strength is replaced by a 7-form [72]. Both these supergravity theories are equivalent and anomaly free when coupled to $SO(32)$ or $E(8) \otimes E(8)$ super-Yang-Mills. For the 3-form formulation the theory is a low energy limit of the heterotic superstrings with $(1,0)$ supersymmetry. The other formulation suggests that there should be a "heterotic" super-5-brane theory dual to the theory of heterotic superstrings [73].

Even more exciting possibility is pointed out by super-2-brane in $D = 11$. It can be checked, that κ symmetry requires that fields g_{MN} and C_{MNR} appearing in (63) satisfy constraints equivalent with the equations of motion of the eleven dimensional supergravity [59, 69]. Moreover double dimensional reduction procedure applied to the super-2-brane gives the type IIA superstring which is S-dual to M-theory. This fact suggests to put forward a hypothesis that M-theory could be a quantum theory of super-2-branes in a similar way as superstring theory is a quantum theory of super-1-branes.

A quantization [74, 71], of super- p -branes for $p > 1$ is significantly more difficult than in the case of superstrings ($p = 1$). The main problem is that for $p > 1$ there are not enough symmetries to gauge away all internal degrees of freedom so in contradistinction to string worldsheet there is no classical meaning of a distance on the worldvolume. Therefore there is no evidence that a brane theory is finite or even renormalizable (but on the other hand neither there is evidence to the contrary). Some attempts to quantize the most promising $p = 2$ brane in $D = 11$ have shown that the resulting theory should be in some aspects similar to super-Yang-Mills theory defined in $D - 1$ dimensions with an exotic gauge group $SU(\infty)$. The result was extended to other super-2-branes and it was also suggested that cases with $p > 2$ correspond to analogs of gauge theory where gauge vectors are replaced by higher rank antisymmetric tensors [75]. Next it was checked that the super-2-brane is anomaly free only in $D = 11$ [77, 76]. However all these studies are rather tests of various possibilities, so M-theory understood as a membrane theory is still more a conjecture than a fact.

3.2 Branes as sources of antisymmetric tensor fields.

In the previous section it was observed that a p -brane is accompanied by a field of $(p+1)$ -forms. This observation is a part of a more fundamental rule: a theory with a $(p+1)$ -form potential is connected with the existence of a $(p+1)$ -dimensional charged objects: branes. Let us examine it more carefully.

Consider a model with an antisymmetric tensor field $A_{[n-1]}$ of rank $n - 1$. A general action of the

model is given by:

$$S = S_{kin} + S_{CS} + S_{int}, \quad (69)$$

$$\begin{aligned} S_{kin} &= \int_{\mathcal{M}_D} \sqrt{g} d^D X e^{a\phi} \left(-\frac{1}{2n!} F^{M_1 \dots M_n} F_{M_1 \dots M_n} \right) \\ &= \int_{\mathcal{M}_D} e^{a\phi} \left(-\frac{1}{2} F_{[n]} \wedge (*F)_{[D-n]} \right) = \int_{\mathcal{M}_D} e^{a\phi} \left(-\frac{1}{2} |F_{[n]}|^2 \right), \end{aligned} \quad (70)$$

where ϕ is a dilatonic scalar field, a – a constant. S_{int} depends at most linearly on $A_{[n-1]}$, but has an arbitrary dependence on any other fields, in particular it has to contain kinetic terms for the dilaton ϕ and graviton g_{MN} . The remaining S_{CS} is Chern–Simons term trilinear in $A_{[n-1]}$. For simplicity of the discussion let us put in this section $S_{CS} = 0$. The F is given by:

$$F_{[n]} = dA_{[n-1]}. \quad (71)$$

With this relation the field F has to satisfy the Bianchi identity:

$$dF_{[n]} = ddA_{[n-1]} = 0. \quad (72)$$

The theory exhibits gauge invariance under:

$$A_{[n-1]} \rightarrow A_{[n-1]} + d\chi_{[n-2]} \quad (73)$$

and the equations of motion derived from (70) are:

$$d(e^{a\phi} (*F)_{[D-n]}) = (*J_e)_{[D-n+1]}, \quad (74)$$

where the $J_{e[n-1]}$ is the conserved Noether current of the theory and its shape is specified by the S_{int} .

This picture is quite similar to that one of electrodynamics but now the current J_e is in general $(n-1)$ -form (and not a vector) and (if it possesses a nonzero timelike component) it defines not a worldline of electrically charged point particle, but $(n-1)$ -dimensional worldvolume of $p = n-2$ dimensional object propagating in time. The object carries an elementary charge of the field F , in other words the Noether charge associated with the conserved Noether current J_e , which by analogy to electrodynamics is very often called an electric charge. The object is known as an electric (elementary) p -brane. If S_{int} has a form of (63) then the p -brane can be identified with the super- p -brane introduced in the section 3.1.

We can calculate the electric charge of the brane in the usual way as:

$$Q_{e[n-1]} = \int_{\mathcal{M}_{D-n+1}} (*J_e)_{[D-n+1]} = \int_{S^{D-n}} e^{a\phi} (*F)_{[D-n]}, \quad (75)$$

where the \mathcal{M}_{D-n+1} is a subspace transversal to the electric brane and the S^{D-n} is a sphere surrounding a point-like image of the brane under a projection of the \mathcal{M}_D on the \mathcal{M}_{D-n+1} .

It is possible to construct also a magnetic (solitonic) brane. Then (71) is replaced by

$$F_{[n]} = dA_{[n-1]} + \eta_{[n]}, \quad (76)$$

where $\eta_{[n]}$ is an arbitrary not exact n -form, so the Bianchi identity is then:

$$dF_{[n]} = (*J_m)_{[n+1]} \quad (77)$$

instead of (72), where $d\eta_{[n]} = (*J_m)_{[n+1]}$. In this way via the modified Bianchi identity a $p = D-n-2$ dimensional object can be introduced to the theory – a magnetic (solitonic) p -brane. The brane carries a magnetic (called also solitonic or topological) charge defined as:

$$Q_{m[D-n-1]} = \int_{\mathcal{M}_{n+1}} (*J_m)_{[n+1]} = \int_{S^n} F_{[n]}, \quad (78)$$

where \mathcal{M}_{n+1} is a subspace transversal to the magnetic brane and the S^n is a sphere surrounding an image of the brane under a projection of \mathcal{M}_D on \mathcal{M}_{n+1} .

When $n = D/2$ both the electric and the magnetic branes have the same dimension and can even coincide. If additionally the field $F_{[n]}$ is selfdual or anti-selfdual, i.e. it obeys:

$$F_{[n]} = \pm * F_{[n]}, \quad (79)$$

the branes necessarily coincide so we then have a single brane carrying simultaneously electric and magnetic charges. This category of branes is called dyonic branes.

Some features of the electric and magnetic branes are strongly correlated. For example:

$$d_e + d_m = D - 2, \quad (80)$$

where d_e and d_m are dimensions of electric and magnetic brane's worldvolumes:

$$d_e = n - 1 \quad d_m = D - n - 1. \quad (81)$$

Because of that it is convenient to define a mapping described by the symbol of tilde \sim and acting on integer numbers as follows:

$$\tilde{d} = D - d - 2. \quad (82)$$

In the case of the brane worldvolume dimensions it gives:

$$\tilde{d}_e = d_m \quad \tilde{d}_m = d_e. \quad (83)$$

Another interesting fact is that if the discussed theory is quantum then the charges (75) and (78) have to satisfy the Dirac's quantization condition:

$$Q_{e[d_e]} Q_{m[d_m]} = 2\pi N, \quad (84)$$

where N is an integer. The above quantization rule was originally derived by Dirac in the case of electromagnetic theory in $D = 4$. To prove (84) in this case consider a hypothetical magnetic monopole (i.e. a magnetically charged particle) with a charge Q_m located at the origin of the coordinate system. Let the coordinates be spherical (r, ϕ, θ) where $r \in (0, \infty)$, $\phi \in [0, 2\pi]$ and $\theta \in [-\pi/2, \pi/2]$. The magnetic monopole is a source of an electromagnetic field whose strength $F_{[2]}$ has to satisfy $\int_{S^2} F_{[2]} = Q_m$, so:

$$F_{[2]} = \frac{Q_m}{4\pi} \cos \theta d\theta \wedge d\phi. \quad (85)$$

Solving $F_{[2]} = dA_{[1]}$ one finds the corresponding vector potential. But the potential cannot be expressed by a single formula globally. It is necessary to introduce at least two maps covering together the whole space with different $A_{[1]}$'s and different choice of gauge on each. For example:

$$A_{[1]}^+ = \frac{Q_m}{4\pi} (\sin \theta + 1) d\phi \quad \text{where } \theta \neq +\pi/2, \quad (86)$$

$$A_{[1]}^- = \frac{Q_m}{4\pi} (\sin \theta - 1) d\phi \quad \text{where } \theta \neq -\pi/2. \quad (87)$$

It gives $(A^+ - A^-)_{[1]} = (Q_m/2\pi)d\phi$. If an electrically charged particle with charge Q_e moves in the field of the magnetic monopole its wave function should also be given by two sections: ψ^+ and ψ^- . Both the wave functions differ by a phase:

$$\exp \left(i \frac{Q_e Q_m}{2\pi} \phi \right). \quad (88)$$

But because shifting the ϕ coordinate by 2π gives the same point, the phase has to be unchanged by such an operation and this requirement leads to $Q_e Q_m = 2\pi N$.

Considering general branes, the Dirac monopole has to be replaced by a magnetic $(D - d - 3)$ -brane and the electric point particle by an electric $(d - 1)$ -brane, where the branes are orthogonal one to the other and not intersecting. Making a projection in the D -dimensional space-time parallel to all space-like directions that are parallel to any of the branes one gets a $(1 + 3)$ dimensional picture identical to the one described before. So, the condition (84) has to be true not only for point particles but also for branes.

It is crucial to observe that if we reformulate the theory defining as a fundamental field $\tilde{F} = *F$, then the equation of motion (74) is replaced by the Bianchi (77) identity and vice versa, but the kinetic term in (70) does not change:

$$F_{[n]} \wedge (*F)_{[D-n]} = (*F)_{[D-n]} \wedge (**F)_{[n]} = \tilde{F}_{[D-n]} \wedge (\tilde{*F})_{[n]}, \quad (89)$$

where the identity $*^2 = (-1)^{1+n(D-n)}$ valid when applied to n -forms was used. This is called electric/magnetic duality. But of course in a general case nothing guarantee that the duality is an exact symmetry of the theory not only an interesting coincidence.

However in 1970s it was noted that in some supersymmetric gauge theories electric and magnetic charges and masses of all particles described by the theory have to obey a universal relation:

$$M^2 = \alpha^2 (Q_e^2 + Q_m^2), \quad (90)$$

where α is a constant. Therefore if roles of the electric and magnetic charges are exchanged the mass is still preserved. This inspired Montonen and Olive to conjecture [78] that the electric/magnetic duality could be a real symmetry of the whole quantum theory. Let us discuss this conjecture in more detail. Consider a quantum state which carries electric and magnetic charges and the electric charge is quantized such that $Q_e = n_e q$, where n_e is an integral quantum number and q is a fundamental quanta of the electric charge. Then the magnetic charge has to be quantized as well, but by the Dirac quantization rule (84) it should be given by $Q_m = n_m (2\pi/q)$. So the electric/magnetic duality cannot be described just by a replacing of the quantum numbers n_e with n_m and vice versa. Simultaneously also the coupling q has to be inversed, what means that the duality connects weak and strong coupled sectors of the theory. This idea has an interesting continuation on the ground of string theory where S-duality was discovered [57, 58].

Up to now we have not specified for what range of p it is reasonable to define a brane. The simple model described in this section works properly if $p = 0, 1, \dots, D - 3$ what corresponds to $d = 1, 2, \dots, D - 2$. For each case there is a well defined antisymmetric tensor potential $A_{[n-1]}$, its strength $F_{[n]}$ and a dual strength $(*F)_{[D-n]}$. In particular, $p = 0$ reduces to a point particle and $p = 1$ – to a string. But it is possible to extend the definition of branes beyond that range.

$p = -1$ makes no formal problems. It is a dual to $(D - 3)$ -brane and is given by a scalar potential and vector strength field. However, some features of the (-1) -brane can seem quite odd. Because its worldvolume is zero-dimensional it cannot propagate and exists only in one moment in time. This class of objects was previously discovered in electrodynamics and are called instantons.

Of course it is possible to extend definition of the instantons, to objects which are not necessary points in spacetime. If we relax the condition, that the currents J_e or J_m must have nonvanishing timelike component, we allow a situation where they span purely spatial hypersurfaces of nonzero dimension. Such hypersurfaces are called S-branes [79, 80, 81]. We will not discuss them in this work and concentrate on the branes evolving with time. It is however worth noting that S-branes can be especially interesting in the context of cosmological models where they can play a role of an initial singularity.

A $(D - 2)$ -brane is usually called a domain wall and it is described by a rank $D - 1$ potential $A_{[D-1]}$ and a rank D strength $F_{[D]}$. Such a field appears for example in massive $(1, 1)$ supergravity (28). Without external sources it can be deduced from equation (74) that $(*F)_{[0]}$ and therefore $F_{[D]}$ have to be constants and have no propagating states. Thus, the kinetic term $|F_{[D]}|^2$ contributes to a lagrangian effectively as a cosmological term.

A brane with $p = D - 1$ is very special. Because its antisymmetric potential has to be proportional to the volume element it should be interpreted just as the whole spacetime. A strength field related to the $(D - 1)$ -brane obviously vanishes since it is a form of rank $D + 1$.

Branes with $p \leq -2$ or $p \geq D$ cannot exist because there is no possibility to introduce an antisymmetric potential in these cases.

3.3 Branes in superstring and M-theory.

Knowing that branes are closely related to antisymmetric forms one may note that the massless sector of every superstring theory contains several fields of this kind. So it is natural to expect that the theories could admit existence of brane-like objects. However there is a question what are these objects in this context and whether they are necessary or only theoretically possible elements of string theories. The answer is more surprising than one could expect. The branes are not only an intrinsic part of any string theory but they also give an excellent tool for forecasting and examining features of the M-theory.

3.3.1 D_p -branes in bosonic strings.

Consider bosonic oriented closed string theory with one, say the twenty sixth, dimension compactified on a circle of radius R . A quantum states' spectrum of such theory is different than the spectrum of free theory (41). However, besides Kaluza-Klein modes which are naturally expected by analogy with the compactification procedure of point particle theories, states of other kind can also appear. They are labelled by so called winding numbers w counting how many times a closed string is wound around the compact dimension. Because closed strings with different w are topologically inequivalent they have to form different states. The full spectrum is:

$$m^2 = \left(\frac{n}{R} + \frac{wR}{\alpha'} \right)^2 + \frac{4}{\alpha'}(N-1), \quad (91)$$

where n numbers Kaluza-Klein excitation levels. A crucial observation is, that the spectrum and the whole theory is invariant under:

$$R \rightarrow R' = \frac{\alpha'}{R}, \quad w \rightarrow n, \quad n \rightarrow w. \quad (92)$$

In other words the theory compactified on a small circle R is equivalent to a theory compactified on a big circle $R' = \frac{\alpha'}{R}$ if roles of the winding and the Kaluza-Klein states are simultaneously interchanged. In the limit $R \rightarrow 0$ a theory dimensionally reduced by one is dual to a theory on a noncompact space $R \rightarrow \infty$. It can be checked that the duality also effectively reverses sign of the right-moving modes, so in terms of coordinates it is given by:

$$\begin{aligned} X^{26}(\xi^1, \xi^2) &= X_L^{26}(\xi^2 - \xi^1) + X_R^{26}(\xi^2 + \xi^1) \rightarrow \\ &\rightarrow X'^{26}(\xi^1, \xi^2) = X_L^{26}(\xi^2 - \xi^1) - X_R^{26}(\xi^2 + \xi^1). \end{aligned} \quad (93)$$

This is an example of T-duality and can be extended to more general cases of toroidal compactifications.

One can wonder what is happening in a similar situation with open strings which cannot have preserved winding numbers. The spectrum of such compactified theory is just:

$$m^2 = \frac{n^2}{R^2} + \frac{4}{\alpha'}(N-1), \quad (94)$$

what evidently is not invariant under $R \rightarrow \frac{\alpha'}{R}$. This seems to lead to a contradiction if one remembers that theories with interacting open strings have to contain also closed strings. The contradiction is however only apparent. A difference between open and closed strings lays in the endpoints of open strings not in their interior. So, a naive consideration gives a prediction, that a dual to open string theory compactified on a circle with radius $R \rightarrow 0$ should be a theory with open strings

having endpoints confined on a 25-dimensional hypersurface. Actually, applying (93) to the Neumann boundary conditions of open string (37) one really obtains Dirichlet boundary conditions:

$$\frac{\partial}{\partial \xi^1} X'^{26}(\xi^1, 0) = 0 = \frac{\partial}{\partial \xi^1} X'^{26}(\xi^1, l), \quad (95)$$

defining a hypersurface called Dirichlet brane, D-brane or D_p -brane, where p is a number of spatial dimensions of the hypersurface [65, 82].

Of course T-duality can be applied not only to the "original" theory where all open strings satisfy only Neumann boundary condition but also to a theory with D_p -brane of arbitrary p . If one T-dualize $k_1 + k_2$ dimensions, k_1 tangent and k_2 orthogonal to the D_p -brane, then one obtains a theory with $D_{(p-k_1+k_2)}$ -brane. In this picture the "original" string theory is a theory with D_{25} -branes filling the whole space.

Additional properties are revealed when one takes into account Chan-Paton states. It can be shown that $U(N)$ oriented open string theory after T-dualization gives a theory with exactly N parallel D_p -branes. A state dual to $|i, j\rangle$ where i, j are indices of the gauge symmetry group is then realized by a string having one end glued to the i -th brane and the second to the j -th brane. If all branes are separated, a gauge symmetry group of the dual theory is $U(1)^N$. If some of the branes coincide, for example if there are n distinct locations for the branes with k_i branes at each ($\sum_{i=1}^n k_i = N$) then the gauge group is $\otimes_{i=1}^n U(k_i)$. Maximally the original $U(N)$ group can be restored.

A little different situation occurs for unoriented strings. The starting symmetry is then $SO(N)$ or $Sp(N)$ and the T-dual space is not compactified on a circle or a torus but rather on an orientifold S^k/\mathbf{Z}_2 . If N is even then all the branes are grouped into $N/2$ pairs with partners living at points related by \mathbf{Z}_2 symmetry, so effectively one sees maximally $N/2$ branes on the orientifold. If N is odd, then there is an additional brane with no partner which has to be localized at \mathbf{Z}_2 fixed plane. A gauge group related to a pair of branes is $U(1)$ and $U(k)$ in a case of k coincident pairs. But if the branes coincide at one of the fixed planes then the symmetry is $SO(2k)$ or $Sp(2k)$ instead of $U(k)$. So in the extreme case when all the branes are at the fixed plane the original symmetry $SO(N)$ or $Sp(N)$ is restored again. Note, that the above results are consistent with the previous observations that a theory with open strings in a flat empty space can be equivalently interpreted as a theory in a space filled out by N D_{25} -branes.

Consider a low energy effective physics on a worldvolume of a single D_p -brane. It has to be given by a vector field $A_\mu(\xi)$ involved with the gauge group of the Chan-Paton states and fields induced on the brane by the background of string massless states:

$$g_{\mu\nu}(\xi) = \frac{\partial X^M}{\partial \xi^\mu} \frac{\partial X^N}{\partial \xi^\nu} g_{MN}(X(\xi)), \quad (96)$$

$$C_{\mu\nu}(\xi) = \frac{\partial X^M}{\partial \xi^\mu} \frac{\partial X^N}{\partial \xi^\nu} C_{MN}(X(\xi)), \quad (97)$$

$$\phi(\xi) = \phi(X(\xi)). \quad (98)$$

As an appropriate action describing a dynamics on the brane one usually postulates Born-Infeld action [83, 84]:

$$S_p = -T_p \int d\xi^{p+1} e^{-\phi(\xi)} \{-\det [g_{\mu\nu}(\xi) + C_{\mu\nu}(\xi) + 2\pi\alpha' F_{\mu\nu}(\xi)]\}^{1/2}, \quad (99)$$

where T_p describes a tension of the brane when an expectation value of the dilaton ϕ vanishes. More precisely a physical tension of the brane in an arbitrary background is:

$$\tau_p = T_p e^{-\langle \phi \rangle}. \quad (100)$$

It is interesting that for D_p branes of various p the following identity holds:

$$\tau_p = \frac{\tau_{p-1}}{2\pi\sqrt{\alpha'}}. \quad (101)$$

The action (99) describes a kind of $(p+1)$ -dimensional gauge theory, where the gauge group and other features of the theory depend directly on the brane configuration and on the type of the string theory. So, we can study gauge theories not only as completely separate models but also in connection with string theory or (in a limit) with supergravity theories.

3.3.2 D_p -branes in superstrings.

Up to now we were considering D-branes in bosonic string theories, but it should be remembered that five consistent, interacting string theories are necessarily supersymmetric. Fortunately, the bosonic sector discussed so far can be simply adjusted – for example one should replace 26 dimensions by 10, set for the tension:

$$\tau_p = \frac{1}{(2\pi)^p e^\phi (\alpha')^{\frac{p+1}{2}}}, \quad (102)$$

add fermionic terms to (99), and then everything what was written for bosonic D-branes is still valid for D-branes in superstring theories. Moreover, new important features can be detected.

Let us focus on the type I $SO(32)$ superstring theory. The theory describes unoriented open and closed strings. After T-dualisation on $9-p$ toroidally compactified dimensions one obtains a theory with open superstrings having endpoints at 16 D_p -branes and closed superstrings propagating in a bulk. There is some evidence that T-duality is an exact symmetry of superstrings, so the dual theory with branes is well defined. Since in a limit when the distance grows to infinity it gives either type IIA or type IIB theory and since changing the distance between two adjacent D-branes is a continuous operation, there are strong conjectures that all the theories at finite distances should also be consistent. In this model the type I, IIA, IIB theories and the theories with the 16 branes are only special limits of more general superstring theory (and further M-theory) and one should rather treat them as different states of the same theory than as separate theories.

It is important to note that similar arguments lead to a prediction that a general superstring state does not need to be given by a set of exactly 16 parallel D-branes of the same dimension. For example one can move to infinity and neglect only some branes but not all of them and get a state with $N < 16$ branes. It is also possible to rotate a brane and then encounter branes intersecting at some angles (even orthogonally in the extreme case). Consider then two orthogonal D_p -branes. Applying T-duality to the configuration in a direction tangent to only one of them one obtains a state with $D_{(p+1)}$ -brane perpendicular to $D_{(p-1)}$ -brane. In a similar way many other states allowing various number of D-branes of various dimensions intersecting at various angles can be constructed.

There is no evidence that all possible states of the string theory have to be built upon a flat space or a space with branes only. Probably other backgrounds also can exist. Unfortunately methods of construction of quantum interacting string theories in a nontrivial background are not yet developed. So, the only possible opportunity to attain a knowledge of the "not flat" states is to study the branes.

In the section 3.2 it was shown that a theory with antisymmetric tensor fields admits an existence of branes – extended charged objects. It was proved that D-branes carry in string theory Ramond-Ramond charges [65], so they are spun by fields from the Ramond-Ramond sector. In the R-R sector of type IIA theory there are antisymmetric tensors $F_{[2]}, F_{[4]}$ and their duals $(*F)_{[6]}$ and $(*F)_{[8]}$. Then the potentials of the fields have ranks 1, 3, 5, 7 and spun D_p -branes of $p = 0, 2, 4, 6$ respectively. In the same way $p = -1, 1, 3, 5, 7$ branes can be related to antisymmetric fields $F_{[1]}, F_{[3]}, F_{[5]} = (*F)_{[5]}, (*F)_{[7]}$ and $(*F)_{[9]}$ of type IIB theory. But there are also D_8 and D_9 -branes which cannot be directly connected with any massless field of the type IIA or the type IIB theory.

The D_8 -brane has to be related to the antisymmetric field strength of rank 10. Such a field has no propagating degrees of freedom in ten dimensions, so including such a field is not obvious from the field theory point of view. But besides the usual $N = (1,1)$ supergravity identified as a low energy limit of the type IIA theory there is also its extension – the massive supergravity (28), which contains the field $F_{[10]}$. On the ground of type IIA superstrings, the existence of D_9 brane can be interpreted as a cosmological constant of arbitrary value [65].

By analogy to other branes one could expect the D₉-brane in type IIB superstrings. The appropriate coupling of the brane to the background fields is:

$$nQ \int A_{[10]}, \quad (103)$$

where n is a number of branes. But the variation of (103) in the action with respect to $A_{[10]}$ leads to $n = 0$. Fortunately in type I superstring with $SO(32)$ gauge symmetry group there are additional terms which modify (103) replacing n by $n - 32$ [85]. So we can interpret the type I theory with its open strings having free ends as a theory defined on 16 coincident pairs of D₉-branes. All the other D-branes can be then derived from the model by T-duality.

3.3.3 BPS and non-BPS, stable and unstable D-brane configurations.

Remembering that five consistent superstring states defined in flat space are supersymmetric and hence stable, it is very interesting to study analogous properties of states with branes.

The type IIA D_{2p}-branes and the type IIB D_{2p+1}-branes always break precisely half of supersymmetries. To see this take a state with 16 parallel branes which is T-dual to type I superstrings. The duality requires that the state must have $N = 1$ supersymmetry. But if a distance between the adjacent branes is growing the state in the bulk tends to type II state which obviously is $N = 2$ supersymmetric. So, half of possible supersymmetries are broken by the branes. Writing more formally, if Q_L and Q_R are supercharges corresponding to left and right moving modes of the theory, then on the D_p-brane:

$$Q_L + \prod_m \beta^m Q_R \quad (104)$$

is conserved, where m denotes p spatial directions perpendicular to the brane and:

$$\beta^m = \Gamma^V \Gamma^m, \quad (105)$$

where Γ^V is the chiral operator in ten dimensions.

The states which preserve part of supersymmetry are usually called BPS-states because they saturate so called Bogomolnyi-Prasad-Sommerfield (BPS) inequality:

$$\mathcal{E} \geq Q, \quad (106)$$

where \mathcal{E} is the energy (mass) density and Q – the charge density. For a single p-brane it can be equivalently written as:

$$\tau_p \geq Q. \quad (107)$$

The formula (104) shows that the number of broken supersymmetries in a given state depends on geometrical properties of configuration of the branes. In particular, an arbitrary number of parallel D_p-branes with the same p breaks the same number of supersymmetries as a single D_p-brane. But if the branes are not parallel, they usually break together more supersymmetries and for some configurations of D-branes no supersymmetry can be preserved at all. However even if two branes are situated at nonzero relative angle it is possible to find such specific value of the angle for which the same amount of supersymmetries is preserved as by the parallel configuration [86].

The non-BPS states also have to be present in string theory. A proof for it is simple. The world we know is nonsupersymmetric, so it has to be a low energy limit of some non-BPS state. An important task consists in finding such a nonsupersymmetric state and test its properties (first of all to check if it is stable).

Probably the simplest example of a non-BPS-state is a system of two parallel D-branes with opposite R-R charges or in other words a system of a D-brane and an anti-D-brane. A key ingredient of the construction is that by (104) each of the objects breaks the complementary half of supersymmetry. In terms of (106) one can see that for the brane-anti-brane system the total mass is twice the mass

of a single brane, but the effective charge vanishes. However such a brane-anti-brane configuration is unstable due to tachyons living on the brane worldvolumes [87].

Another non-BPS-state can be produced from a system of coincident D_{2p} -brane and anti- D_{2p} -brane in type IIA theory with the projection operator $(-1)^{F_L}$ acting on it [88]. The $(-1)^{F_L}$ changes sign of the left moving fermions, so it brings the type IIA superstrings in the bulk to the type IIB. It also removes half of states on the brane-anti-brane system, specifically this half which is related to degrees of freedom describing possibility of disjoining the branes. So, the object is effectively a single D_{2p} -brane in the type IIB theory. An analogous construction leads to type IIA D_{2p+1} -branes. Both the classes of branes are unstable and their masses are by $\sqrt{2}$ bigger than masses of corresponding type IIA or IIB BPS-branes:

$$\tau_{2p,IIB} = \sqrt{2}\tau_{2p,IIA}, \quad \tau_{2p+1,IIA} = \sqrt{2}\tau_{2p+1,IIB}. \quad (108)$$

Applying the $(-1)^{F_L}$ to the non-BPS type IIB D_{2p} -brane or type IIA D_{2p+1} -brane once again changes the bulk theory and projects out next part of states living on the brane. The result of the operation is the already known BPS IIA D_{2p} -brane or IIB D_{2p+1} -brane respectively.

But we are still looking for the stable non-BPS states. They are interesting for several reasons:

- They belong to the string theory spectrum, so they are necessary to fully describe the theory.
- By (106) they are objects whose masses are not bounded by demand of supersymmetry, but still possible to calculate for various values of the string perturbative coupling constant. Therefore they provide an opportunity to study string theory at finite coupling.
- The worldvolume theory on the stable non-BPS brane should belong to nonsupersymmetric gauge theories which are much less understood than the supersymmetric ones.

The usual procedure leading to construction of a stable non-BPS brane is to apply to one of IIA or IIB unstable non-BPS states an orbifolding or orientifolding projection which removes the tachyonic states [88]. In this way, acting with the worldsheet parity operator Ω on the IIB D_0 -brane the type I stable D_0 -brane can be found. This example is very interesting because it can serve as an illustration to the S-duality between type I and heterotic $SO(32)$ theories. At the first massive level the heterotic theory possesses non-BPS states which are in the spinor representation of the $SO(32)$ gauge group. But because these states are the lightest in such representation they could not decay without violating the quantum numbers conservation law. Hence, they are stable. Now one can identify the states as the dual partners of the stable type I D_0 -branes.

The stable non-BPS brane states can be also studied in type IIA theory compactified on orbifold T^4/\mathcal{I}^4 , where \mathcal{I}^4 is the spacetime parity which changes signs of the four compactified coordinates. The model contains for example D_1 -branes. By T-dualisation it gives type IIB theory on $T^4/(-1)^{F_L}\mathcal{I}^4$ which is furthermore dual to the IIB on $T^4/\Omega\mathcal{I}^4$. For more examples see also [88, 89, 90].

3.3.4 NS-branes.

There is one more antisymmetric field, common for massless limits of all five basic superstring theories and not correlated to any of the D_p -brane. This is the rank three tensor $H_{[3]}$ belonging to the NS-NS sector as it was shown in (11), (19) and (27). The antisymmetric potential of the field couples to the fundamental superstring, so we can interpret the superstring as an electric NS_1 -brane.

Furthermore an existence of a magnetic NS_5 -brane, can be predicted. Such an object is a solitonic solution of classical equations of motion. It will be shown later that there is also an additional argument based on S-duality that the magnetic NS-branes should be an intrinsic part of the general superstring theory.

First consider a fundamental string and a D_1 -brane in type IIB theory. The objects are similar but not identical. Both are 1-branes and both have the same massless quantum excitations, but their tensions are different and obey the relation:

$$\frac{\tau_{F1}}{\tau_{D1}} = e^\phi. \quad (109)$$

The above relation shows that in the weakly coupled limit where $e^\phi \ll 1$ the fundamental string is much lighter than the D₁-brane but this is no longer true when the coupling is getting stronger. For a strongly coupled case with $e^{\phi'} = e^{-\phi}$ the situation is opposite. The symmetry of inverting the coupling constant with simultaneous exchange of F-string and D₁-brane is called weak-strong duality and is a simplest example of S-duality. So, both states should be thought of in general as different manifestations of the same object. Similarly it can be checked that type IIB D₃-brane is S-selfdual.

Now let us take D₅-brane in type IIB theory. It is electric/magnetic dual to the D₁ brane, which is S-dual to the fundamental string, which is in turn electric/magnetic dual to the NS₅-brane. Therefore to preserve consistency, D₅-brane and NS₅-brane should be related by S-duality:

$$\begin{array}{ccc} F_1 & \leftarrow S \rightarrow & D_1 \\ \uparrow & & \uparrow \\ el/mag & & el/mag \\ \downarrow & & \downarrow \\ NS_5 & \leftarrow S \rightarrow & D_5 \end{array} \quad (110)$$

Recall that a fundamental open string has its ends attached to D_p-branes. But if one object is joined with another, their duals have to be tied up by the same relation. Therefore if we have F-string with its endpoints on a D₅-brane, we should expect also a D₁-brane glued to NS₅-brane. Analogously from a composition of a F-string and a D₃-brane, a system where a D₁-brane ends on a D₃-brane can be deduced. Applying to the system various T-dualities one obtains a D_p-brane joined to D_q-brane where p and q are arbitrary.

3.3.5 M-branes.

Even more spectacular results follow when S-duality acts on objects in the type IIA theory, especially on D₀-branes. Because the D_p-brane sweeps out a $p + 1$ dimensional worldvolume and its tension is given by (102), the mass scale related to the D_p-brane is of order:

$$m_p \approx (e^{-\phi})^{\frac{1}{p+1}} \alpha'^{-1/2}. \quad (111)$$

Therefore at strong coupling the lightest states are built of the D₀-branes and total mass of a system consisting n objects of this type is equal to:

$$\frac{n}{e^\phi \alpha'^{1/2}}. \quad (112)$$

If the coupling e^ϕ tends to infinity, a split between states with n and $(n + 1)$ branes tends to zero and in the limit the spectrum becomes continuous. This picture looks identically as a model with the eleventh dimension compactified on a circle of radius:

$$R_{11} = e^\phi \alpha'^{1/2}, \quad (113)$$

when the radius grows up. This is the essence of the argument that the M-theory S-dual to type IIA should be eleven dimensional.

The M-theory is largely unknown, but because we assume that its low energy limit is described by $D = 11$ supergravity, we can predict some properties of it. In particular, because in the action (5) a third rank antisymmetric potential is present, we expect an existence of so called M₂ and M₅-branes carrying respectively electric and magnetic charges of $A_{[3]}$. Furthermore we can describe the branes of type IIA superstrings in terms of M-theory with compactified eleventh dimension:

- F-string is the M₂-brane wrapped on the eleventh dimension,
- NS₅-brane is the M₅-brane transverse to the compactified dimension,
- D₀-brane is a state carrying Kaluza-Klein electric charge,

- D₂-brane is the M₂-brane transverse to the compactified dimension,
- D₄-brane is the M₅-brane wrapped on the eleventh dimension,
- D₆-brane is a Kaluza-Klein magnetic monopole,

Summarizing these considerations one can conjecture that in the unified theory there should be essentially only one type of electric brane and one type of magnetic brane.

4 Branes in supergravity.

Most of supergravity theories have in its field content one or more antisymmetric tensors. Therefore as observed in the previous section 3.2, one should expect that among possible classical solutions there exist also brane solutions. Such solutions can be found by at least two equivalent methods. The first one consists in solving the system of equations with an inhomogeneity given by a delta function (source term in (63)) coinciding with the location of a brane. The second method consists in solving the homogenous equations of motion and imposing boundary conditions appropriate to given brane configuration. Integration constants (or their combinations) describing the solution should be identified with physical quantities like tension or charge.

The problem of finding and classifying all (or at least as many as possible) classical solutions for a given model is of fundamental importance not only because these solutions characterize the basic features of the model but also because they serve as the building blocks for constructing Hilbert space in quantum theory. The importance of the issue in the case of higher dimensional supergravities is strongly amplified since they are identified as the low energy limit of the respective superstring states (which are largely unknown except in the simplest cases). One can therefore expects that supergravity description of the branes should be instructive also from the superstring theory point of view. First because it gives information about the underlying classical geometry and second because supergravity solutions should give us insight into the superstring spectrum. Furthermore to solve the supergravity equations of motion exact methods can be used while in superstring theory only perturbative methods are known. Therefore the results obtained in these two ways can be in some aspects complementary.

There are however several problems which have to be stressed. Supergravity is only an effective low energy limit of the superstring theory, so there is no guarantee that the exact solution derived with constraints describing a given brane configuration is still a good classical approximation of any exact solution in the quantum theory. It has to be checked case by case whether the constraints imposed agree with conditions leading to the low energy limit [91, 92]. More precisely, the supergravity action is the limit of the string theory where $\alpha' \rightarrow 0$. Because a tension of a p-brane (102) is of order $\alpha'^{-(p+1)/2}$, the relation between a given string state and its low energy (supergravity) description should be well defined when the total mass of the respective brane configuration goes to infinity. In the case of BPS branes, due to vanishing force theorem, one can prove that a solution describing n branes is a direct superposition of n single brane solutions with the total mass:

$$M_{n,p} = n^{p+1}\sqrt{\tau_p}, \quad (114)$$

which tends to infinity for $n \rightarrow \infty$. But this simple picture usually does not work for non-BPS states and despite some partial results [93, 94], the question of classical description of non-BPS states is still open.

Another problem has a technical character. While there are mathematical theorems stating that functions satisfying the equations of motion with given boundary conditions always exist, they do not provide us these solutions in explicit form. In other words, we are able to find only a limited number of solutions, usually only if some additional assumptions leading to simplification of the starting equations are made. Therefore each essentially new solution is a significant achievement.

4.1 Single charge solution in the harmonic gauge.

Let us consider at the beginning one of the simplest models (in detail discussed in [95]): D -dimensional theory describing graviton g_{MN} , dilaton ϕ and a single antisymmetric potential $A_{[n-1]}$ with its field strength $F_{[n]}$. We examine only a consistent bosonic truncation of the theory, i.e. with no terms with fermionic fields taken into account. Equivalent formulation of the condition is an assumption that vacuum values of the fermionic fields are identically equal to zero. Additionally the contribution from the Chern-Simons term is neglected.

The action is then:

$$S = \int_{\mathcal{M}} dX^M \sqrt{|g|} \left(R - \partial_M \phi \partial^M \phi - \frac{1}{2} e^{a\phi} |F_{[n]}|^2 \right) \quad (115)$$

and the equations of motion:

$$R_{MN} = \frac{1}{2}\partial_M\phi\partial_N\phi + S_{MN}, \quad (116)$$

$$0 = \nabla_M(e^{a\phi}F^{N_1\dots N_n}), \quad (117)$$

$$\nabla^2\phi = \frac{a}{2}e^{a\phi}|F_{[n]}|^2, \quad (118)$$

where:

$$S_{MN} = \frac{1}{2}e^{a\phi}\left(\frac{1}{(n-1)!}F_{[n]MR_1\dots R_{n-1}}F_{[n]N}{}^{R_1\dots R_{n-1}} - \frac{n-1}{D-2}|F_{[n]}|^2g_{MN}\right). \quad (119)$$

We search for a single charge solution, what means that the $A_{[n-1]}$ has only one independent non-zero component which carries electric or magnetic charge but not both (the case of dyonic brane is excluded). The brane enforces a split of the whole space-time \mathcal{M} into a multiplication of two mutually orthogonal subspaces ¹ $V \otimes V_\emptyset$, where V is a worldvolume of the brane and $V_\emptyset = \mathcal{M}/V$. We assume that the solution is maximally space-time symmetric what is equivalent to breaking by the brane of the Poincaré symmetry $ISO(D-1, 1)$ to subgroup $ISO(d-1, 1) \times SO(D-d)$ where d is the dimension of the brane worldvolume.

Under these assumptions we can write the following ansätze for the metric tensor and the dilaton:

$$ds^2(X) = e^{2A(r)}dx^\mu dx^\nu \eta_{\mu\nu} + e^{2B(r)}dy^m dy^n \delta_{mn}, \quad (120)$$

$$\phi(X) = \phi(r), \quad (121)$$

where x^μ with $\mu = 1, \dots, d$ are coordinates in directions parallel to the brane, y^m with $m = d+1, \dots, D$ – in transversal and:

$$r = \sqrt{y^m y^n \delta_{mn}}. \quad (122)$$

For the antisymmetric tensor we need to distinguish two separate cases. The electric brane corresponds to F with the only nonvanishing component:

$$F_{m\mu_1\dots\mu_d}(X) = \sigma\epsilon_{\mu_1\dots\mu_d}\partial_m \exp(C(r)), \quad (123)$$

where $\sigma = \pm 1$, but the solitonic brane:

$$F_{m_1\dots m_{\tilde{d}+1}}(X) = \epsilon_{m_1\dots m_{\tilde{d}+1}n} \frac{\lambda y^n}{r^{\tilde{d}+2}}, \quad (124)$$

where λ is a real constant.

Substituting (120 – 124) into (116 – 118) we can rewrite the equations of motion as:

$$A'' + d(A')^2 + \tilde{d}A'B' + \frac{\tilde{d}+1}{r}A' = \frac{\tilde{d}}{2(D-2)}(S')^2, \quad (125)$$

$$B'' + \tilde{d}(B')^2 + dA'B' + \frac{2\tilde{d}+1}{r}B' + \frac{d}{r}A' = -\frac{d}{2(D-2)}(S')^2, \quad (126)$$

$$\tilde{d}B'' + dA'' - 2dA'B' + d(A')^2 - \tilde{d}(B')^2 - \frac{\tilde{d}}{r}B' - \frac{d}{r}A' + \frac{1}{2}(\phi')^2 = \frac{1}{2}(S')^2, \quad (127)$$

$$\phi'' + dA'\phi' + \tilde{d}B'\phi' + \frac{\tilde{d}+1}{r}\phi' = -\frac{\zeta a}{2}(S')^2, \quad (128)$$

$$C'' + C'\left(C' + \frac{\tilde{d}+1}{r} - dA' + \tilde{d}B' + a\phi'\right) = 0, \quad (129)$$

¹For simplicity we identify a submanifold with its tangent bundle.

where the prime denotes a derivation with respect to r and:

$$\varsigma = \begin{cases} +1 & (\text{electric}), \\ -1 & (\text{magnetic}), \end{cases} \quad (130)$$

$$S' = \begin{cases} \sigma \exp(\frac{1}{2}a\phi - dA)(e^C)' & (\text{electric}), \\ \exp(\frac{1}{2}a\phi - \tilde{d}B) \frac{\lambda}{r^{d+1}} & (\text{magnetic}). \end{cases} \quad (131)$$

A crucial observation is that the equations are drastically simplified when additional assumptions about the solution are made:

$$dA' + \tilde{d}B' = 0, \quad (132)$$

$$\tilde{d}\phi' + \varsigma a(D-2)A' = 0, \quad (133)$$

where:

$$\tilde{d} > 0, \quad a \neq 0. \quad (134)$$

The (132) is sometimes called the harmonic gauge because it always leads to solutions where the metric tensor is expressed via harmonic functions [96, 97]. In this case, the solution reads:

$$ds^2 = H(r)^{\frac{-4\tilde{d}}{\Delta(D-2)}} dx^\mu dx^\nu \eta_{\mu\nu} + H(r)^{\frac{4d}{\Delta(D-2)}} dy^m dy^n \delta_{mn}, \quad (135)$$

$$e^\phi = H(r)^{\frac{2\varsigma a}{\Delta}}, \quad (136)$$

$$F^{electric}_{m\mu_1 \dots \mu_d} = \sigma \epsilon_{\mu_1 \dots \mu_d} \partial_m (H(r)^{-1}), \quad (137)$$

$$F^{magnetic}_{m_1 \dots m_{\tilde{d}+1}} = \epsilon_{m_1 \dots m_{\tilde{d}+1} n} \partial^n H(r), \quad (138)$$

$$H(r) = 1 + \frac{k}{r^d}, \quad (139)$$

$$\Delta = \frac{2d\tilde{d}}{D-2} + a^2, \quad (140)$$

where k is a positive integration constant – in the magnetic case:

$$k = \frac{\sqrt{\Delta}\lambda}{2\tilde{d}}. \quad (141)$$

To arrive at a more symmetric relation between electric and magnetic cases, it is convenient to use this identity as a definition of λ for the electric brane. Note, that the constant Δ can be equivalently written as:

$$\Delta = \frac{2}{\frac{1}{d} + \frac{1}{\tilde{d}}} + a^2, \quad (142)$$

so it is just harmonic average of d and \tilde{d} enlarged by a^2 . Setting $\phi = 0$ and $a = 0$ in the above solution we obtain a solution for a model without the dilaton. If we include a dilaton but still keep $a = 0$ it is necessary to replace (136) by:

$$e^\phi = H(r). \quad (143)$$

In some special cases, the solution (135-143) describes:

- M₂-brane [98] if $D = 11$, $d = 3$ and $\phi = 0$,
- M₅-brane [99] if $D = 11$, $d = 6$ and $\phi = 0$,
- NS₁-brane [100] if $D = 10$, $d = 2$ and $a = -1$,
- NS₅-brane [101] if $D = 10$, $d = 6$ and $a = -1$.

4.1.1 Geometry of the solution.

The positivity of k of the previous subsection is a consequence of an extra requirement that we want to avoid a singularity for $r > 0$. Other restrictions imposed on integration constants to derive the above solutions are: $A, B, \phi \rightarrow 0$ when $r \rightarrow \infty$, what means that at the infinity the solution asymptotically goes to Minkowski spacetime. Near $r = 0$ the geometry is curved and corresponds to $AdS_{d+1} \times S^{d+1}$.

Consider the solution without the dilaton. More careful analysis shows that the point $r = 0$ is rather a horizon than a singularity and the solution can be extended beyond it [102, 103, 104]. It is convenient to introduce the so called interpolating coordinates where r is replaced by r_{int} satisfying:

$$r^{\tilde{d}} = \frac{kr_{int}^d}{1 - r_{int}^d}. \quad (144)$$

So, the "flat infinity" $r \rightarrow \infty$ corresponds to $r_{int} \rightarrow 1$ and the horizon $r = 0$ to $r_{int} = 0$. The horizon is sometimes called a degenerate one, because its properties are not the same as for the horizon in the classical Schwarzschild solution. Particularly for the brane solution the g_{tt} component of the metric tensor does not change its sign at the horizon as it happens at the Schwarzschild horizon. In other words light-cones cannot flip over inside the horizon.

The interpolating coordinates are well defined also for $r_{int} < 0$. If d is odd it can be checked that at $r_{int} \rightarrow -\infty$ the solution describes a geometry near a naked singularity and the singularity can be therefore identified with a localization of a fundamental brane. If d is even, the formula (144) is invariant under $r_{int} \leftrightarrow -r_{int}$, so the area described by negative r_{int} is a mirror of the area where r_{int} is positive and there is no singularity at all. Of course the construction is only one of many possible analytical continuations. However its virtue is that in $D = 11$ supergravity it allows to identify the electric 2-brane with the singular solution as it should be for the elementary object of the theory, while the magnetic non-singular 5-brane solution can be interpreted as a soliton.

If we consider the solution with the dilaton the situation is a little different. The field ϕ is scalar, so it is invariant under any coordinate change and the points where it tends to infinity are necessary singularities of the whole solution. By (136) it means that the horizon $r = 0$ coincides with the singularity.

The solution (135-143) can be extended to cases $\tilde{d} = 0$ or $\tilde{d} = -1$ when the factor $k/r^{\tilde{d}}$ is replaced by $k \ln r$ or kr respectively. But for such solutions it is not true that at $r \rightarrow \infty$ they describe Minkowski spacetime or (if $\tilde{d} = 0$) that there is no singularity of the metric for positive r .

4.1.2 Charges and tension.

Let us calculate a number of degrees of freedom of the solution (135 – 140). The ansätze were expressed in terms of four scalar function A, B, C, ϕ which appear in the equations of motion (125 – 129) with their second derivatives. Integrating the equations, one should expect 8 integration constants. But, since there are four function and five equations, one of them can be used as a constraint reducing the number of free parameters by one. The linearity conditions (132 – 133) cancel next two and the requirement of $A, B, \phi \rightarrow 0$ at $r \rightarrow \infty$ additional three. One of the remaining two illustrates the fact that $A_{[n-1]}$ is determined up to an additive constant. The last one is the parameter k and being the only essential parameter it has to determine both the energy and charge of a solution.

We can calculate the electric and magnetic charges associated with the solution from the identities (75) and (78). If we use the relation (141) then we get:

$$Q_{e/m} = \lambda \Omega_{\tilde{d}+1} = \frac{2k\tilde{d}}{\sqrt{\Delta}} \Omega_{\tilde{d}+1}, \quad (145)$$

where $\Omega_{\tilde{d}+1}$ is a surface of unit $(\tilde{d} + 1)$ -sphere.

Next, thanks to the ADM mass formula which reads [105, 106]:

$$g_{tt} = -1 + \frac{\tau_{d-1}}{\Omega_{\tilde{d}+1}(D-2)r^{\tilde{d}}} + o\left(\frac{1}{r^{\tilde{d}}}\right), \quad \text{when } r \rightarrow \infty, \quad (146)$$

or equivalently [95]:

$$\tau_{d-1} = \int_{\partial V_0} d\Omega_{\tilde{d}+1} r^{\tilde{d}} y^m (\partial^n h_{mn} - \partial_m h_i^i), \quad (147)$$

where:

$$g_{MN} = \eta_{MN} + h_{MN} \quad (148)$$

and the indices i run over all $D-1$ spatial directions, we find the tension (or the energy density) of the solution:

$$\tau_{d-1} = \frac{2}{\sqrt{\Delta}} \lambda \Omega_{\tilde{d}+1}. \quad (149)$$

Another way leading to the same result is to find the stress-energy density pseudotensor:

$$T_{MN} = T(A)_{MN} + T(\phi)_{MN}, \quad (150)$$

where $T(A)_{MN}$ and $T(\phi)_{MN}$ are contribution to the total stress-energy density coming from the antisymmetric potential and the dilaton respectively. From (116) we see, that:

$$T(A)_{MN} = S_{MN} - \frac{1}{2} S_R^R g_{MN} = \frac{e^{a\phi}}{2(n-1)!} F_{[n]MR_1\dots R_{n-1}} F_{[n]N}{}^{R_1\dots R_{n-1}}, \quad (151)$$

$$T(\phi)_{MN} = \frac{1}{2} \partial_M \phi \partial_N \phi - \frac{1}{4} \partial_R \phi \partial^R \phi g_{MN}. \quad (152)$$

And the energy density corresponding to the T_{tt} component of T_{MN} , is:

$$T_{tt} = T(A)_{tt} + T(\phi)_{tt} = (S')^2 + (\phi')^2. \quad (153)$$

The energy of the intersecting branes system is then:

$$\mathcal{E} = \int_{\mathcal{M}} d^D X \sqrt{|\det g|} T(A)_t^t = \Omega_{\tilde{d}+1} |V| \int_0^\infty dr r^{\tilde{d}+1} (S')^2 = \frac{2\lambda}{\sqrt{\Delta}} \Omega_{\tilde{d}+1} |V| \quad (154)$$

and it is precisely equal to (149) multiplied by the brane worldvolume.

In $D=11$ supergravity the electric brane solution is characterized by $d=3$, $\tilde{d}=6$, $a=0$ and the magnetic one by $d=6$, $\tilde{d}=3$, $a=0$. So, for both $\Delta=4$. At $D=10$ NS-branes have $d=2$, $\tilde{d}=6$ or $d=6$, $\tilde{d}=2$ and $a=-1$ what gives $\Delta=4$ again. Also, the analogous solutions related to D-branes have the same value of Δ . Therefore all of them saturate the BPS bound (106) and are supersymmetric. Also in lower dimensions most antisymmetric fields and then the branes corresponding to them are described by $\Delta=4$ so the solutions in harmonic gauge preserve half of the original supersymmetries. This general rule is a consequence of a fact that a value of the Δ is preserved under Kaluza-Klein dimensional reduction [107] and all supergravity theories in lower dimensions can be derived with this procedure from $D=11$ supergravity or type $N=2$, $D=10$ supergravity. However other values of Δ are also possible. It can happen when the antisymmetric tensor in the considered supergravity theory is a linear combination of several fields of the same rank obtained by the dimensional reduction procedure [108]. Such solution observed from the higher dimensional point of view describes rather several coinciding branes.

4.1.3 Supersymmetry.

The supersymmetry preserving condition is in general given by the requirement that for all fermionic fields Ψ in the theory the vacuum value of its supersymmetric transformation $\langle \delta_\eta \Psi \rangle$ is vanishing. We take as an example an electric 2-brane in $D = 11$ supergravity. Because the theory possesses only one fermionic field Ψ_M , the supersymmetry preserving condition is just:

$$\delta_\eta \Psi_M = 0, \quad (155)$$

where $\delta_\eta \psi_M$ is given by (8). The elementary brane splits the 11-dimensional spacetime into $3 + 8$, hence 11-dimensional Dirac matrices Γ_M have to decompose as:

$$\Gamma_M \rightarrow (e^A \Gamma_\mu, e^B \Gamma_m), \quad (156)$$

$$\Gamma_\mu = \gamma_\mu \times \Sigma_V, \quad (157)$$

$$\Gamma_m = Id \times \Sigma_m, \quad (158)$$

where γ_μ are Dirac matrices in 3 dimensions, Σ_m — Dirac matrices in 8 dimensions and Σ_V is the 8-dimensional chiral operator. It is convenient to introduce:

$$\Gamma_V = Id \times \Sigma_V, \quad (159)$$

so the following identities hold:

$$\Gamma^{\mu\nu\rho} \epsilon_{\mu\nu\rho} = 6\Gamma_V, \quad (160)$$

$$\Gamma^{\mu\nu} \epsilon_{\mu\nu\rho} = 2\Gamma_\rho \Gamma_V. \quad (161)$$

Substituting (120) and (123) into (155) we get:

$$\delta_\eta \Psi_\mu = \frac{1}{6} e^{-2A-B} \Gamma_\mu \Gamma^m \partial_m (e^{3A} + \sigma e^C \Gamma_V) \eta, \quad (162)$$

$$\delta_\eta \Psi_m = \left(\partial_m N + \frac{\sigma}{6} e^{C-3A} \partial_m C \Gamma_V \right) \eta + \frac{1}{2} \Gamma_m^n \left(\partial_n B - \frac{\sigma}{6} e^{C-3A} \partial_n C \Gamma_V \right) \eta. \quad (163)$$

where:

$$\eta(r) = e^{N(r)} \eta_0 \quad (164)$$

and η_0 is a constant parameter. So, we see, that the condition (155) is satisfied only if simultaneously:

$$2B' + A' = 0, \quad (165)$$

$$\frac{d}{dr} e^{3A} = \sigma \frac{d}{dr} e^C, \quad (166)$$

$$2N' = A', \quad (167)$$

$$\Gamma_V \eta_0 = -\sigma \eta_0. \quad (168)$$

The first of these conditions is equivalent to (132) and the second is a consequence of (132) and (133). This shows explicitly that harmonic gauge is a necessary condition for preserving supersymmetry. The third condition states that the preserved supersymmetry is rigid and the fourth that the spinorial parameters corresponding to the preserved part of the supersymmetry are chiral in the 8 dimensions transversal to the brane. We see also that (168) is invariant under a transformation:

$$\sigma \rightarrow -\sigma \quad (169)$$

describing a reversal of the electric charge and a reversal of the parameter η chirality, at once. In other words, the brane and anti-brane preserve opposite chirality parts of supersymmetry.

For a magnetic 5-brane in the $D = 11$ supergravity an equivalent derivation to the previously described can be conducted giving analogous conclusions. However, in this case instead of (162 – 163) one has:

$$\delta_\eta \Psi_\mu = \frac{1}{2} e^{A-B} \Gamma_\mu \Gamma^m \frac{y_m}{r} \left(A' + \frac{1}{6} e^{-3B} \frac{\lambda}{r^4} \Gamma_V \right) \eta, \quad (170)$$

$$\delta_\eta \Psi_m = \frac{y_m}{r} \left(N' + \frac{1}{12} e^{-3B} \frac{\lambda}{r^4} \Gamma_V \right) \eta + \Gamma_m^n \frac{y_m}{r} \left(\frac{1}{2} B' - \frac{1}{6} e^{-3B} \frac{\lambda}{r^4} \Gamma_V \right) \eta, \quad (171)$$

with:

$$\Gamma^{mnrst} \epsilon_{mnrst} = 120 \Gamma_V. \quad (172)$$

The decomposition of the Dirac matrices gives now:

$$\Gamma_\mu = \gamma_\mu \otimes \Sigma_{11}, \quad (173)$$

$$\Gamma_m = Id \otimes \Sigma_m, \quad \text{for } m = 7, \dots, 10, \quad (174)$$

$$\Gamma_{11} = \gamma_V \otimes \Sigma_{11}, \quad (175)$$

where γ_μ ($\mu = 1, \dots, 6$) are the Dirac matrices defined on the worldvolume of the brane with γ_V the six dimensional chiral operator constructed of them and Σ_m ($m = 7, \dots, 11$) are the Dirac matrices on the remaining five dimensions. So, for the operator Γ_V we have:

$$\Gamma_V = \gamma_V \otimes Id. \quad (176)$$

4.1.4 Multi center solution.

The crucial trick applied to find the solution (135 – 140) was to make additional assumptions simplifying the equations of motion (116 – 118) which allowed to reduce it to:

$$\nabla^2 H = \left(\partial_r^2 + \frac{\tilde{d}+1}{r} \partial_r \right) H = 0. \quad (177)$$

But because the operator ∇^2 is linear it means that if any two H_1 and H_2 obey (177) their sum $H_1 + H_2$ is also a good solution of the equation. Therefore in general one can replace (139) by:

$$H(r) = 1 + \sum_{i=1}^N \frac{k_i}{|\vec{y} + \vec{y}_{0i}|^{\tilde{d}}}. \quad (178)$$

A natural interpretation of this so-called multi-center solution is to assume that it describes a set of N parallel identically oriented (i.e. characterized by the same sign of the R-R charge) branes, each localized at $y^m = y_{0i}^m$. Physically, a possibility of such a solution follows from a fact that in such configuration of branes attractive forces carried by the graviton g_{MN} and the dilaton ϕ are precisely cancelled by repulsive forces of identically oriented antisymmetric fields. The whole brane configuration breaks the same amount of supersymmetry as each of its components and the total charge density and energy density is given by:

$$Q = \Omega_{\tilde{d}+1} \sum_i \lambda_i, \quad \mathcal{E} = \Omega_{\tilde{d}+1} \frac{2}{\sqrt{\Delta}} \sum_i \lambda_i. \quad (179)$$

4.2 Nonsupersymmetric single-charge solutions.

The solution given in the previous section can be generalized if some of the constraints imposed on the model are relaxed. One possibility is to drop the harmonic gauge condition (132). A class of solutions obtained in this way was initially discussed in [109, 110] and a complete solution was

presented in [111]. We will discuss those cases in more detail in chapter 5. Let us only mention here that since harmonic gauge is not imposed, the equations of motion cannot be reduced to (177) and the solutions are not still governed by harmonic functions and are nonsupersymmetric in general. In a connection with string theory they can be used as the classical description of the brane-anti-brane system [112].

4.2.1 Black branes.

Another possibility is to relax the demand of the $ISO(d - 1, 1)$ symmetry on the brane world-volume. Branes of this type are called black branes [99, 91, 101, 114] because of their similarity to the Schwarzschild black hole². The simplest case occurs when instead of the $ISO(d - 1, 1)$ we have $SO(d - 1)$ symmetry. The appropriate ansatz for the metric tensor is then:

$$ds^2(X) = -e^{2A_t(r)}dt^2 + e^{2A_x(r)}dx^{\tilde{\mu}}dx_{\tilde{\mu}} + e^{2B(r)}dy^m dy_m, \quad (180)$$

where the indices $\tilde{\mu}$ run through the $d - 1$ spatial directions parallel to the brane. In such a case the harmonic gauge condition (132) has to be rewritten in the form:

$$A'_t + (d - 1)A'_x + \tilde{d}B' = 0. \quad (181)$$

It is convenient to write the black brane solution in Schwarzschild coordinates, where the Schwarzschild radial coordinate r_s is related to the original isotropic r by:

$$r = r_s \left(\frac{\sqrt{H_+(r_s)} + \sqrt{H_-(r_s)}}{2} \right)^{\frac{2}{d}}, \quad (182)$$

$$H_{\pm}(r_s) = 1 - \left(\frac{r_{\pm}}{r_s} \right)^d. \quad (183)$$

The spacetime interval takes then a form:

$$\begin{aligned} ds^2 &= H_-(r_s)^{\frac{4\tilde{d}}{\Delta(D-2)}} \left(-\frac{H_+(r_s)}{H_-(r_s)} dt^2 + dx^{\tilde{\mu}} dx_{\tilde{\mu}} \right) \\ &\quad + H_-(r)^{\frac{2a^2}{\Delta d}-1} \left(\frac{1}{H_+(r_s)H_-(r_s)} dr_s^2 + r_s^2 d\Omega_{d+1}^2 \right) \end{aligned} \quad (184)$$

and the scalar field:

$$e^\phi = H_-(r_s)^{\frac{2sa}{\Delta}}. \quad (185)$$

The solution depends on two nonnegative parameters r_+ and r_- . The first describes a localization of an event horizon and the second of an inner horizon. Both the horizons are nongenerate, it means that similarly as for the horizon in the Schwarzschild black hole solution signs of the g_{tt} and g_{rr} components of the metric tensors are reversed when one goes with the solution through the horizon. If $r_+ = r_-$ both the sing reversions cancel one with the other and one obtains the degenerate horizon known for the supersymmetric brane solution discussed in the paragraph 4.1.1. The horizon at $r_s = r_+$ is nonsingular, but the horizon at $r_s = r_-$ usually coincides with a singularity what is a consequence of a fact that the dilaton depends on H_- .

The two parameters r_+ and r_- describe also the energy density and the charge density of the black brane:

$$\mathcal{E} = \Omega_{\tilde{d}+1} \tilde{d} (r_+ r_-)^{\tilde{d}/2}, \quad (186)$$

$$Q = \Omega_{\tilde{d}+1} \left((\tilde{d}+1)r_+^{\tilde{d}} - r_-^{\tilde{d}} \right). \quad (187)$$

²More precisely a black hole can be interpreted as a black 0-brane.

So, we see that in general the solution does not saturate the BPS bound and is not supersymmetric in spite of the fact that it is expressed in terms of harmonic functions H_+ and H_- . This explicitly illustrates that the harmonic gauge leads to harmonic functions in the solution but it is not sufficient to preserve supersymmetry. However in a special situation when $r_+ = r_-$ the solution becomes supersymmetric and reproduces the isomorphic solution (135 – 140).

Finally it is also possible to find a brane solution without imposing the harmonic gauge and Poincaré worldvolume symmetry. This kind of generalized black branes was given in [115] and discussed in [112].

4.3 Solutions with many branes.

The procedure of relaxing some constraints imposed on the equations of motion gives a very rich collection of various solutions if we allow for configurations describing many branes instead of the single brane. Such configurations can be introduced to the model in two fundamental ways (with possible combinations of the two).

The first is to consider a theory with several, say N_A , antisymmetric tensors $F_{[n_i]}^i$ and assume that each of the fields still supports only one brane. Because the ranks n_i can take various values, the branes have dimensions potentially completely uncorrelated with one another. An action relevant for such model can be built starting from the single brane action (115) by replacing the single kinetic terms of antisymmetric fields by a sum:

$$\frac{1}{2} \sum_{i=1}^{N_A} e^{a_i \phi} \left| F_{[n_i]}^i \right|^2. \quad (188)$$

In the equations of motion instead of S_{MN} we obtain a sum $\sum_i S_{MN}^i$ where each component describes the contribution from different $F_{[n_i]}^i$ and thus different branes. The system of equations is also enlarged because (117) now appears in N_A copies, one for each $F_{[n_i]}^i$.

In the second method we still have only one antisymmetric tensor but allow it to be the source of several branes. In this case the branes are of only two possible kinds: one electric and one magnetic. The action of the model is the same as (115), all differences are encoded in the form of the assumed solution. Of course a natural generalization is to combine these two ways and study brane configurations related to systems with many antisymmetric fields each describing many independent branes. The situation where many branes are given by one antisymmetric field is known in literature as composite branes.

The composite branes of a single antisymmetric tensor can be in some cases equivalently described in terms of several single charged antisymmetric fields. This possibility arises if the first component of the S_{MN} (119) tensor has diagonal form or equivalently when the stress–energy tensor $T(A)_{MN}$ (151) is diagonal. Then:

$$F_{[n]MR_1\dots R_{n-1}} F_{[n]N}{}^{R_1\dots R_{n-1}} = 0, \quad \text{if} \quad M \neq N. \quad (189)$$

This means that there are no direct interactions among independent elements of the $F_{[n]}$ [116].

In supergravity theories there can also exist different scalar fields so the multi-scalar brane models should be also considered. But in supergravities there are two kinds of scalar fields each having different features. One type consists of dilaton fields which in lagrangians appear as exponential factors multiplying the kinetic terms of the antisymmetric fields. The other type consists of rank zero antisymmetric potentials which can be responsible for the existence of instantonic branes.

For a single brane it is always possible to choose Cartesian coordinate system with d directions parallel to the brane and $\tilde{d} + 2 = D - d$ perpendicular to it and describe localization of the brane by constraints $X^m = X_0^m$ where m runs from $d + 1$ to D . This possibility is very convenient because it allows to identify the brane worldvolume with a single independent element of $A_{[n-1]}$ antisymmetric potential. But for two or more branes the feature has in general no simple extension. If we orient the Cartesian coordinate system to be in an agreement with one brane nothing can a priori guarantee

that the other is either parallel or perpendicular to the directions of the coordinate system. In other words the branes can be oriented at any angle one to the other.

But in this work we restrict ourselves only to configurations of orthogonally or parallelly oriented branes. Consider then two branes of this kind: a p_1 -brane and a p_2 -brane. Such a configuration induces a split of the spacetime directions into four segments: $\{1, 2\}$, $\{1\}$, $\{2\}$ and \emptyset . The $\{1, 2\}$ contains directions parallel to both of the branes. It is necessarily not trivial because the time always belongs to it. The $\{1\}$ segment is characterized as tangent to the first and perpendicular to the second brane. Analogously the $\{2\}$ segment – parallel to the second but transverse to the first. Finally the \emptyset segment contains directions normal to both the branes. The directions in the $\{1, 2\}$ segment are usually called common tangent, in the $\{1\}$ and $\{2\}$ – relative transverse and in the $\{\emptyset\}$ – overall transverse. Further we can decompose the set of the coordinates $\{X^M\}$ into four subsets:

$$\{X^M\} \rightarrow \{x^{\mu_{\{1,2\}}}, x^{\mu_{\{1\}}}, x^{\mu_{\{2\}}}, y^m\}. \quad (190)$$

A spacetime localization of the branes is completely determined when the following constraints are imposed:

$$\begin{aligned} x^{\mu_{\{2\}}} &= x_1^{\mu_{\{2\}}}, & y^m &= y_1^m, & \text{for the first brane,} \\ x^{\mu_{\{1\}}} &= x_2^{\mu_{\{1\}}}, & y^m &= y_2^m, & \text{for the second brane.} \end{aligned} \quad (191)$$

where $x_2^{\mu_{\{1\}}}, x_1^{\mu_{\{2\}}}, y_1^m, y_2^m$ are constants. The splitting procedure can be extended to arbitrary N_A and always gives one common tangent segment, no more than one overall transverse and maximally $2^{N_A} - 2$ relative transverse segments.

4.3.1 Intersecting branes.

The two branes as described before intersect orthogonally³ when $y_1^m = y_2^m$ and the intersection is a subspace given by:

$$x^{\mu_{\{1\}}} = x_2^{\mu_{\{1\}}}, \quad x^{\mu_{\{2\}}} = x_1^{\mu_{\{2\}}}, \quad y^m = y_1^m = y_2^m. \quad (192)$$

A description of branes intersections becomes more complicated when the number of branes is bigger than two because intersections of any two different pairs of branes may have no common point. However, in a special situation there exists nonempty subspace common for all the N_B branes. It is naturally to name the case as commonly intersecting branes however it is usually called just "intersecting branes" [119] what is shorter but less precise. The (commonly) intersecting brane configurations are very special and have many features distinguishing them from the others, what is a consequence of a fact that they have relatively more spacetime symmetries. Let us discuss it in some detail.

Each p -brane breaks the original $ISO(D-1, 1)$ symmetry of flat empty spacetime to $ISO(p, 1) \times SO(D-p-1)$. But when we have two or more branes the symmetry is usually only a local approximation, because in globally each brane can break a different part of $ISO(D-1, 1)$. Therefore a group of the commonly preserved symmetry is much smaller and in the extreme case can be reduced to a translation in time exclusively. But the situation is different if the branes are commonly orthogonally intersecting because then the preserved symmetry has always a form:

$$ISO(d_{\{1, 2, \dots, N_A\}} - 1, 1) \times \left(\bigotimes_{\tilde{I}} SO(d_{\tilde{I}}) \right) \quad (193)$$

where $d_{\{1, 2, \dots, N_A\}}$ is a dimension of the intersection common for the all N_A branes and $d_{\tilde{I}}$ are dimensions of the other segments of the spacetime distinguished by the branes.

For a solution corresponding to a brane configuration which is characterized by (193) it is natural to assume that it depends on variables $r_{\tilde{I}}$, where $r_{\tilde{I}}$ is the radial coordinate in the \tilde{I} -th segment and where the origin of the coordinate system lies on the common intersection. But when such ansätze are

³Of course in general branes can intersect not only orthogonally but also at arbitrary angles. See for example [86, 117, 118]

imposed the equations of motion form a quite complicated system of partial differential equations with second order derivatives with respect to the $r_{\tilde{I}}$'s. It is not easy to solve that system in full generality. Additional restrictions simplifying the equations are needed.

Usually it is assumed that the solution has to depend only on r – radial coordinate in the overall transverse space. This kind of solutions are called delocalized, averaged or smeared branes, because they do not determine at what points the branes are situated in the relative transverse directions. One can wonder if the solution derived under such assumption can have any physical meaning. It turns out that there is one important application – if the relative transverse dimensions are small in comparison to the common tangent and overall transverse ones since then we can use the intersecting branes configurations as a background for compactification models (with the relative transverse dimensions compactified). We have to note that the notion of intersection of delocalized branes has a rather imprecise meaning – if they are not localized we cannot be sure that they really intersect. But because such terminology is common, we will use it here too.

Since a set of orthogonally intersecting branes enforces a split of the spacetime into several orthogonal segments it is natural to assume that an ansatz for the metric tensor should admit independent factors to encode possibly different length scales for each of the segments. Call $e^{A_{\{1,\dots,N_A\}}}$, e^{A_I} , e^B the factors for the common tangent, the relative transverse and the overall transverse directions respectively. This dramatically increases the number of degrees of freedom of the solution and complicates its derivation. Therefore one usually imposes some set of linearity conditions as in the case of the single brane. One of them is the generalized harmonic gauge:

$$d_{\{1,2,\dots,N_A\}} A_{\{1,\dots,N_A\}} + \sum_{\tilde{I}} d_{\tilde{I}} A_{\tilde{I}} + \tilde{d} B = 0, \quad (194)$$

where

$$\tilde{d} = D - d_{\{1,2,\dots,N_A\}} - \sum_{\tilde{I}} d_{\tilde{I}} - 2. \quad (195)$$

Applying it as an additional requirement on the solution has analogous effects as for the single brane – it enforces the solution to be constructed of harmonic functions only and is necessary but not sufficient for preserving the supersymmetry. It is also possible to impose more restrictive conditions:

$$A_{\tilde{I}} = A_{\tilde{J}} \quad \text{for all } \tilde{I}, \tilde{J}, \quad (196)$$

what means that all relative transverse dimensions are governed by the same factor, or even:

$$A_{\tilde{I}} = A_{\{1,\dots,N_A\}} \quad \text{for all } \tilde{I}. \quad (197)$$

The multibrane systems are classified as BPS or non-BPS states where the criterion is respectively saturation or non-saturation of the multibrane version of the BPS inequality (106). But while the total energy is just a sum of component energies:

$$\mathcal{E} = \sum_i \mathcal{E}_i, \quad (198)$$

the charge is in general rather "vector-like":

$$Q^2 = \sum_i Q_i^2. \quad (199)$$

The BPS states fall into two categories: marginal and non-marginal [120]. For the marginal (called also threshold) states the BPS bound is degenerated and can be written as:

$$\mathcal{E} = \sum_i Q_i. \quad (200)$$

Examples of intersecting branes solutions which satisfy (194) are solutions constructed as orthogonal superposition of some number of the single charge supersymmetric brane solutions (135 – 140) with the use of so called harmonic function rule [119, 121, 122]. The rule for a nondilatonic solution tells, that if:

$$H_i(r) = 1 + \frac{k_i}{r^d}, \quad (201)$$

for $i = 1, \dots, N_A$, and D_i is a dimension of the i -th brane worldvolume, then:

$$e^{2A_{\{1, \dots, N_A\}}} = \prod_{i=1}^{N_A} H_i^{\frac{D_i}{D-2}-1}, \quad (202)$$

$$e^{2A_{\tilde{I}}} = \prod_{i \in \text{tan}(\tilde{I})} H_i^{\frac{D_i}{D-2}-1} \prod_{j \in \text{trans}(\tilde{I})} H_j^{\frac{D_j}{D-2}}, \quad (203)$$

$$e^{2B} = \prod_{i=1}^{N_A} H_i^{\frac{D_i}{D-2}}, \quad (204)$$

where $\text{tan}(\tilde{I})$ (respectively $\text{trans}(\tilde{I})$) is a set of such indices i which correspond to the branes for which the \tilde{I} -th segment of the metric contains directions tangent (transversal) to the brane worldvolume.

Wider class of solutions can be obtained when the multibrane equations of motion are solved directly. It is interesting that it is possible to reduce the equations of the intersecting branes to a Toda-like system which is an extension of the Liouville equation. A possibility of the reduction was observed in [109] under assumption of (196), in generalized harmonic gauge [123, 124] and further it was proved in a general case without any linearity condition assumed [125]. We will analyze the last case with details in the next chapter. But of course all the examples given already and later do not run short a set of possible but still solvable generalizations of the brane problem. For some others see [119, 95, 126, 97, 127].

5 Branes without harmonic gauge.

In the chapter 4 a short introduction to the subject of branes in supergravity was given. It was noted that in the case of single brane as well as in the case of many intersecting branes a rich class of nonsupersymmetric solution could be derived when the condition of the harmonic gauge was dropped. Now we turn our attention to this kind of solutions and to a method leading to them.

5.1 Commonly orthogonally intersecting non-composite delocalised branes.

Consider a D dimensional theory having after consistent bosonic truncation a following action:

$$S = \int_{\mathcal{M}} d^D X \sqrt{|\det g|} \left(R - \frac{1}{2} \sum_{\alpha=1}^{N_\phi} \partial_M \phi_\alpha \partial^M \phi_\alpha - \sum_{i=1}^{N_A} \frac{e^{\sum_{\alpha=1}^{N_\phi} a_{i\alpha} \phi_\alpha}}{2} |F^i|^2 \right), \quad (205)$$

where F^i are antisymmetric n_i -forms, ϕ^α – scalar fields, $a_{i\alpha}$ – constants, \mathcal{M} – a manifold of dimension D and (X^M) coordinates on it. A bosonic sector of most of supergravity theories like (5), (11), (27) is well described by the above action if additional assumptions leading to cancellation of Chern-Simons term are made.

The equations of motions derived from (205) are:

$$R_{MN} = \frac{1}{2} \sum_{\alpha} \partial_M \phi_\alpha \partial_N \phi_\alpha + \sum_i \frac{e^{\sum_{\alpha} a_{i\alpha} \phi_\alpha}}{2(n_i - 1)!} S_{MN}^i, \quad (206)$$

$$0 = \nabla_M \left(e^{\sum_{\alpha} a_{i\alpha} \phi_\alpha} F^{iMR_1 \dots R_{n_i-1}} \right), \quad (207)$$

$$\nabla^2 \phi_\alpha = \sum_i \frac{a_{i\alpha}}{2n_i!} e^{\sum_{\beta} a_{i\beta} \phi_\beta} F_{R_1 \dots R_{n_i}}^i F^{iR_1 \dots R_{n_i}}, \quad (208)$$

with:

$$S_{MN}^i = F_{MR_1 \dots R_{n_i-1}}^i F_N^{iR_1 \dots R_{n_i}} - \frac{n_i - 1}{n_i(D-2)} F_{R_1 \dots R_{n_i}}^i F^{iR_1 \dots R_{n_i}} g_{MN}, \quad (209)$$

where \sum_i and \sum_{α} are sums over all possible values of $i = 1, \dots, N_A$ and $\alpha = 1, \dots, N_\phi$.

5.1.1 The model.

We search for a solution which allows N_A commonly orthogonally intersecting electric or magnetic non-composite branes. Let V_i be a worldvolume of the i -th brane. Then each V_i is supported by a potential of a different F^i or $*F^i$. We define indices I, J, \dots running through the set of all non-empty subsets of $\{1, \dots, N_A\}$ and V_I with $I = \{i_1, \dots, i_k\}$ as a subspace spun by vectors simultaneously parallel to all V_{i_1}, \dots, V_{i_k} and transversal to all $V_{i_{k+1}}, \dots, V_{i_{N_A}}$. Next, let V be a cartesian product of common tangent and all relative transverse directions, V_\emptyset – the overall transverse space and \hat{V}_i – a subspace of V transverse to V_i . Volumes of the subspaces V_i , V_I , \hat{V}_i will be denoted respectively by $|V_i|$, $|V_I|$ and $|\hat{V}_i|$. If all V_i intersect at the point $\{0\}$ (the center of the coordinate system) we can write:

$$V_I = \{0\} \cup \left(\left(\bigcap_{i \in I} V_i \right) \setminus \left(\sum_{j \notin I} V_j \right) \right), \quad (210)$$

$$V = \sum_i V_i = \oplus_I V_I, \quad (211)$$

$$V_\emptyset = \mathcal{M}/V, \quad (212)$$

$$\hat{V}_i = \sum_{I: i \notin I} V_I = V/V_i. \quad (213)$$

Introduce numbers describing dimensions of the subspaces:

$$d_I = \dim V_I, \quad (214)$$

$$D_i = \dim V_i = \sum_{I:i \in I} d_I, \quad (215)$$

$$\hat{D}_i = \dim \hat{V}_i = \sum_{I:i \notin I} d_I, \quad (216)$$

$$d = \dim V = \sum_I d_I. \quad (217)$$

For particular brane configurations some of V_I can be zero-dimensional. For example, if $V_i \subset V_j$ for some i, j , then $V_{\{i\}} = \{0\}$. We will use also the mapping $\tilde{}$ defined by (82).

All the branes are assumed to propagate in time, so the instantonic solution is a priori disallowed. Note that because the branes are delocalized in the relative transverse directions, the solution would be trivial giving all fields constant when $V_\emptyset = \{0\}$. So we not consider the case. Therefore, the dimensions (214 – 217) have to obey:

$$1 \leq d_{\{1, \dots, N_A\}} \leq D - 1, \quad (218)$$

$$-1 \leq \tilde{d} \leq D - 3, \quad (219)$$

$$0 \leq d_I \leq D - 2 \text{ for } I \neq \{1, \dots, N_A\}. \quad (220)$$

5.1.2 Ansätze.

Let us now set ansätze consistent with all assumed conditions. We call by (x^{μ^i}) the coordinates on V_i , by $(x^{\hat{\mu}^i})$ – the coordinates on \hat{V}_i , by (x^{μ^I}) – on V_I , by (y^m) – on V_\emptyset and we use sum convention for all indices enumerating coordinates but not for indices like i or I . Because we search for a delocalized solution all the fields should depend nontrivially only on $r = \sqrt{y^m y^n \delta_{mn}}$ – a radial coordinate in the overall transverse space. Derivatives with respect to the r are denoted by primes $f' = df/dr$.

The simplest is an ansatz for the scalar fields:

$$\phi_\alpha(X) = \phi_\alpha(r). \quad (221)$$

More complicated is a case of the metric tensor. It has to be divided into N_g segments related to V_\emptyset and those V_I which are at least one-dimensional ($2 \leq N_g \leq 2^{N_A}$):

$$ds^2(X) = \sum_I e^{2A_I(r)} dx^{\mu^I} dx^{\nu^I} \eta_{\mu^I \nu^I} + e^{2B(r)} \left((dr)^2 + r^2 d\Omega_{(\tilde{d}+1)}^2 \right), \quad (222)$$

where $d\Omega_{(\tilde{d}+1)}^2$ is the space interval of the $\tilde{d} + 1$ dimensional unit sphere and $\eta_{\mu^I \nu^I} = \delta_{\mu^I \nu^I}$ if $I \neq \{1, \dots, N_A\}$. It has to be explained that in the formula (222) and any formulae below by $\sum_{I:r(I)}$ we denote a sum over those I for which V_I are at least one-dimensional and which satisfy the restriction $r(I)$.

For the antisymmetric tensor fields F^i , two cases should be distinguished. If the i -th brane is electric, the only nonzero components of F^i have the form:

$$F_{m\mu_1^i \dots \mu_{D_i}^i}^i(X) = \sigma_i \epsilon_{\mu_1^i \dots \mu_{D_i}^i} \partial_m \exp(C_i(r)), \quad (223)$$

when for the i -th brane being magnetic only:

$$F_{\hat{\mu}_1^i \dots \hat{\mu}_{D_i}^i m_1 \dots m_{\tilde{d}+1}}^i(X) = \epsilon_{\hat{\mu}_1^i \dots \hat{\mu}_{D_i}^i m_1 \dots m_{\tilde{d}+1} n} \frac{\lambda_i y_n}{r^{\tilde{d}+2}} \quad (224)$$

do not vanish. In the above λ_i is an arbitrary nonzero real constant and σ_i takes discrete values $+1$ or -1 . In the both cases the relative opposite signs illustrate possibility of an exchanging in the model the given brane with its anti-brane.

Now it is possible to derive explicit forms of the Ricci tensor and other objects appearing in the equations of motion. But first we need to find a vielbein, which is defined as:

$$e_{M\bar{N}} e_{R\bar{S}} \eta^{\bar{N}\bar{S}} = g_{MR}, \quad (225)$$

$$e_{M\bar{N}} e_{R\bar{S}} g^{MR} = \eta_{\bar{N}\bar{S}}, \quad (226)$$

where the bared indices enumerate the local Lorentz coordinates which are lifted by $\eta^{\bar{M}\bar{N}}$. So in our case from (222) we have:

$$e_M^{\bar{M}} = \begin{pmatrix} \text{diag}_I \left(\exp(A_I(r)) e_{\mu^I}^{\bar{\mu}^I} \right) & 0 \\ 0 & \exp(B(r)) e_m^{\bar{m}} \end{pmatrix}, \quad (227)$$

where

$$e_{M\bar{N}} e_{R\bar{S}} \eta^{\bar{N}\bar{S}} = \eta_{MR}, \quad e_{M\bar{N}} e_{R\bar{S}} \eta^{MR} = \eta_{NS}. \quad (228)$$

With the vielbein we can calculate a spin connection $\omega_{M\bar{N}\bar{R}}$ from:

$$\omega_{M\bar{N}\bar{R}} = \frac{1}{2}(e_N^S \Omega_{MS\bar{R}} - e_{\bar{R}}^S \Omega_{MS\bar{N}} - e_{\bar{N}}^S e_{\bar{R}}^T e_M^{\bar{U}} \Omega_{ST\bar{U}}), \quad (229)$$

$$\Omega_{MN\bar{R}} = \partial_M e_{N\bar{R}} - \partial_N e_{M\bar{R}}. \quad (230)$$

We find that the only nonzero components of the field Ω are:

$$\Omega_{\mu^I n\bar{\rho}^I} = -(\partial_n A_I) e^{A_I} e_{\mu^I \bar{\rho}^I}, \quad (231)$$

$$\Omega_{mn\bar{r}} = e^B ((\partial_m B) e_{n\bar{r}} - (\partial_n B) e_{m\bar{r}}), \quad (232)$$

so the spin connection is described as:

$$\omega_{\mu^I \bar{\nu}^I \bar{r}} = e^{A_I - B} (\partial_{\bar{r}} A_I) e_{\mu^I \bar{\nu}^I}, \quad (233)$$

$$\omega_{m\bar{n}\bar{r}} = (\partial_{\bar{r}} B) e_{m\bar{n}} - (\partial_{\bar{n}} B) e_{m\bar{r}} \quad (234)$$

and has all other elements vanishing.

The Riemann tensor can be derived from the spin connection because it obeys:

$$R_{MNRS} = 2e_{R\bar{R}} e_{S\bar{S}} \left(\partial_{[M} \omega_{N]}^{\bar{R}\bar{S}} + \omega_{[M}^{\bar{R}\bar{T}} \omega_{N]}^{\bar{T}\bar{S}} \right), \quad (235)$$

so in our case its nonzero components are given as:

$$R_{\mu^I \nu^I \rho^I \sigma^I} = e^{4A_I - 2B} (A'_I)^2 (\eta_{\mu^I \sigma^I} \eta_{\nu^I \rho^I} - \eta_{\mu^I \rho^I} \eta_{\nu^I \sigma^I}), \quad (236)$$

$$\begin{aligned} R_{\mu^I n\rho^I s} = & -e^{2A_I} \left[\left(A''_I + (A'_I)^2 - 2A'_I B' - \frac{1}{r} A'_I \right) \frac{y_n y_s}{r^2} + \right. \\ & \left. + \left(A'_I B' + \frac{1}{r} A'_I \right) \delta_{ns} \right] \eta_{\mu^I \nu^I}, \end{aligned} \quad (237)$$

$$R_{\mu^I \nu^J \rho^I \sigma^J} = -e^{2A_I + 2A_J - 2B} A'_I A'_J \eta_{\mu^I \rho^I} \eta_{\nu^J \sigma^J}, \quad \text{where } I \neq J, \quad (238)$$

$$\begin{aligned} R_{mnrs} = & e^{2B} \left[\left((B')^2 + \frac{2}{r} B' \right) (\delta_{nr} \delta_{ms} - \delta_{mr} \delta_{ns}) + \right. \\ & \left. + \left(B'' - (B')^2 - \frac{1}{r} B' \right) \left(\frac{y_m y_s}{r^2} \delta_{nr} + \frac{y_n y_r}{r^2} \delta_{ms} - \frac{y_m y_r}{r^2} \delta_{ns} - \frac{y_n y_s}{r^2} \delta_{mr} \right) \right]. \end{aligned} \quad (239)$$

The Ricci tensor can be calculated from:

$$R_{NS} = R_{MNRS} g^{MR} \quad (240)$$

and all its elements are zero except:

$$R_{\mu^I \nu^I} = -e^{2(A_I - B)} \left(A''_I + \frac{\tilde{d}+1}{r} A'_I + \sum_J d_J A'_J A'_I + \tilde{d} A'_I B' \right) \eta_{\mu^I \nu^I}, \quad (241)$$

$$\begin{aligned} R_{mn} &= - \left[B'' + \frac{2\tilde{d}+1}{r} B' + \tilde{d}(B')^2 + \sum_I d_I A'_I \left(B' + \frac{1}{r} \right) \right] \delta_{mn} + \\ &\quad - \left[\tilde{d} B'' - \frac{\tilde{d}}{r} B' - \tilde{d}(B')^2 + \sum_I d_I \left(A''_I - 2A'_I B' + (A'_I)^2 - \frac{1}{r} A'_I \right) \right] \frac{y_m y_n}{r^2}. \end{aligned} \quad (242)$$

And finally the Ricci scalar:

$$\begin{aligned} R &= R_{MNG}^{MN} \\ &= -e^{-2B} \left[2 \sum_I d_I \left(A''_I + (A'_I)^2 + A_I \sum_J d_J A'_J + \frac{2(\tilde{d}+1)}{r} A'_I + 2\tilde{d} A'_I B' \right) + \right. \\ &\quad \left. + \frac{2(\tilde{d}+1)^2}{r} B' + \tilde{d}(\tilde{d}+1)(B')^2 + 2(\tilde{d}+1)B'' \right]. \end{aligned} \quad (243)$$

Let us rewrite the equations of motion (206–208) in terms of the functions A_I , B , ϕ_α and (only for the elementary branes) C_i introduced by (221–224):

$$A''_I + A'_I \left(\sum_J d_J A'_J + \tilde{d} B' + \frac{\tilde{d}+1}{r} \right) = \frac{\sum_{i \in I} \tilde{D}_i (S'_i)^2 - \sum_{i \notin I} D_i (S'_i)^2}{2(D-2)}, \quad (244)$$

$$B'' + \tilde{d}(B')^2 + \frac{2\tilde{d}+1}{r} B' + (B' + \frac{1}{r}) \sum_I d_I A'_I = -\frac{\sum_i D_i (S'_i)^2}{2(D-2)}, \quad (245)$$

$$\phi''_\alpha + \phi'_\alpha \left(\sum_I d_I A'_I + \tilde{d} B' + \frac{\tilde{d}+1}{r} \right) = -\frac{1}{2} \sum_i \varsigma_i a_{i\alpha} (S'_i)^2, \quad (246)$$

$$\begin{aligned} \tilde{d} B'' - \tilde{d}(B')^2 - \frac{\tilde{d}}{r} B' + \sum_I d_I \left(A''_I - \frac{1}{r} A'_I - 2A'_I B' + (A'_I)^2 \right) &= \\ &= \frac{1}{2} \sum_i (S'_i)^2 - \frac{1}{2} \sum_\alpha (\phi'_\alpha)^2, \end{aligned} \quad (247)$$

$$C'_i \left(C'_i - \sum_{I:i \in I} d_I A'_I + \sum_{I:i \notin I} d_I A'_I + \tilde{d} B' + \sum_\alpha a_{i\alpha} \phi'_\alpha + \frac{\tilde{d}+1}{r} \right) = -C''_i, \quad (248)$$

where:

$$\varsigma_i = \begin{cases} +1 & (\text{electric}), \\ -1 & (\text{magnetic}), \end{cases} \quad (249)$$

$$S'_i = \begin{cases} \sigma_i (e^{C_i})' \exp(\frac{1}{2} \sum_\alpha a_{i\alpha} \phi_\alpha - \sum_{I:i \in I} d_I A_I) & (\text{electric}), \\ \frac{\lambda_i}{r^{\tilde{d}+1}} \exp(\frac{1}{2} \sum_\alpha a_{i\alpha} \phi_\alpha - \sum_{I:i \notin I} d_I A_I - \tilde{d} B) & (\text{magnetic}). \end{cases} \quad (250)$$

5.1.3 Harmonic gauge.

Assuming that $\tilde{d} \neq 0$ define a function:

$$\chi = \sum_I d_I A_I + \tilde{d} B. \quad (251)$$

Condition $\chi = 0$ is equivalent to the generalized harmonic gauge (194). Imposing it enormously simplifies the equations (244–248) and leads to a solution expressed in terms of harmonic functions on V_\emptyset [97, 96]. In case of supergravity theories the condition is necessary but not sufficient for preserving supersymmetry. Here we do not make any a priori assumption on χ , so the results presented below remain valid in more general classes of nonsupersymmetric solutions and solutions not governed by harmonic functions.

Summing (244–245) one can see that χ has to satisfy the following equation:

$$\chi'' + (\chi')^2 + \frac{2\tilde{d}+1}{r}\chi' = 0, \quad (252)$$

which can be solved and gives:

$$\chi(r) = \ln \left| \frac{c_\chi - 1/r^{2\tilde{d}}}{c_\chi - c_0} \right| + \epsilon_\chi(c_0), \quad (253)$$

where c_χ and c_0 are constants taking real as well as infinite values and ϵ_χ is a function of c_0 . Of course it is only one of many possible variants how the χ can be parameterized. But with this one we have:

$$\lim_{c_\chi \rightarrow +\infty} \chi = \lim_{c_\chi \rightarrow -\infty} \chi = \epsilon_\chi(c_0). \quad (254)$$

So points $c_\chi = +\infty$ and $c_\chi = -\infty$ can be identified in the parameter space and then the space is compact in c_χ direction. Harmonic gauge is restored when $c_\chi = \pm\infty$ and $\epsilon_\chi = 0$.

See, that because of arbitrariness of ϵ_χ the parameterization used in (253) is not unique but rather constitutes a class of parameterizations labelled by c_0 . Each choice of c_0 gives different parameterization, but each is singular at $c_\chi = c_0$. In the discussion below we choose $c_0 = 1$, but we should remember that it can be generalized to arbitrary c_0 .

It is very convenient to introduce instead of c_χ new parameters: $R \in [0, +\infty]$ and $s_\chi \in \{-1, +1\}$ such that:

$$s_\chi R^{2\tilde{d}} = 1/c_\chi, \quad (255)$$

so if $R = 0$ or $R = \infty$ both possible signs of s_χ describe the same point in the parameter space. Rewriting χ with these parameters one obtains:

$$\chi(r) = \ln \left| \frac{1 - s_\chi (R/r)^{2\tilde{d}}}{1 - s_\chi R^{2\tilde{d}}} \right| + \epsilon_\chi. \quad (256)$$

For $R = 0$ (harmonic gauge) it simplifies to $\chi = \epsilon_\chi$ and for $R = \infty$ to $\chi = -2\tilde{d} \ln r + \epsilon_\chi$.

5.1.4 ϑ coordinate.

Looking at the equations of motion (244–248) one can see that the equations with second derivative of A_I or ϕ_α have a form:

$$f'' + \left(\chi' + \frac{\tilde{d}+1}{r} \right) f' = \text{const}(S')^2. \quad (257)$$

The left hand side is just $\nabla^2 f$, so (257) is curved space harmonic equation where a contribution from the curvature is given by χ' . Under a redefinition of variable as $r \rightarrow \vartheta$ (257) transforms to:

$$\ddot{f}(\vartheta')^2 + \left(\vartheta'' + \chi' \vartheta' + \frac{\tilde{d}+1}{r} \vartheta' \right) \dot{f} = \text{const}(\dot{S}\vartheta')^2, \quad (258)$$

where the "dots" describe derivatives with respect to ϑ . And this convinces that it would be very convenient to work with such variable ϑ for which:

$$\vartheta'' + \chi' \vartheta' + \frac{\tilde{d}+1}{r} \vartheta' = 0. \quad (259)$$

A function which satisfies the above equation is:

$$\vartheta(r) = \begin{cases} \frac{1}{2\tilde{d}} \left(\frac{1}{R^{\tilde{d}}} + R^{\tilde{d}} \right) \left(\arctan\left(\left(\frac{R}{r}\right)^{\tilde{d}}\right) - \arctan(R^{\tilde{d}}) \right), & s_\chi = -1, \\ \frac{1}{2\tilde{d}} \left| \frac{1}{R^{\tilde{d}}} - R^{\tilde{d}} \right| \left(\text{Arth}\left(\left(\frac{R}{r}\right)^{\tilde{d}}\right) - \text{Arth}(R^{\tilde{d}}) \right), & s_\chi = +1, \end{cases} \quad (260)$$

where:

$$\text{Arth}(x) = \begin{cases} \text{ar tanh}(x), & \text{for } |x| < 1, \\ -\text{ar coth}(x), & \text{for } |x| > 1. \end{cases} \quad (261)$$

Of course this is not the most general solution of (259) which can be generated from the function ϑ given in (260) as $a\vartheta + b$ where the integration constants a and b can be independently set in two areas separated by points where χ is singular. One of the areas is $r < R$, $s_\chi = +1$ and the second is the remaining part of the spacetime.

The function ϑ obeys following conditions:

$$\lim_{R \rightarrow \infty} \vartheta(r; R, s_\chi = \pm 1) = -\frac{1}{2\tilde{d}} \left(r^{\tilde{d}} - 1 \right), \quad (262)$$

$$\lim_{R \rightarrow 0} \vartheta(r; R, s_\chi = \pm 1) = \frac{1}{2\tilde{d}} \left(\frac{1}{r^{\tilde{d}}} - 1 \right). \quad (263)$$

So in harmonic gauge ($R = 0$) and only then ϑ is a flat space harmonic function of r . And the parameter R (or c_χ equivalently) can be then treated as a measure how distant is a given case from the harmonic one.

The function ϑ can be understood as a space coordinate instead of r and the coordinate change is singular only at $r = R$, $s_\chi = +1$. Further, in the section 5.3 we will see, that the replacing of r with ϑ is not only a simple coordinate change, because the solution formulated with use of ϑ is well defined also for such values of ϑ which cannot be related to any r by (260). In other words, ϑ covers wider area of spacetime than r .

The space-time interval expressed in terms of ϑ is:

$$ds^2(\vartheta) = \sum_I e^{2A_I(\vartheta)} dx^{\mu^I} dx_{\mu^I} + e^{2B_\vartheta(\vartheta)} (d\vartheta^2 + \rho(\vartheta)^2 d\Omega^2), \quad (264)$$

where

$$e^B = e^{B_\vartheta} |\vartheta'|, \quad (265)$$

$$\rho = |\vartheta'| r \quad (266)$$

and the coordinate change factor ϑ' because of (260) has to be equal to:

$$\frac{d\vartheta}{dr} = -\frac{1}{2r^{\tilde{d}+1}} \exp(-\chi(r) + \epsilon_\chi). \quad (267)$$

The identity (266) allows us to write ρ explicitly in terms of the variable r or ϑ :

$$\rho(r) = \begin{cases} \frac{1}{2} \left| \frac{(1/R)^{\tilde{d}} + R^{\tilde{d}}}{(r/R)^{\tilde{d}} + (R/r)^{\tilde{d}}} \right|, & s_\chi = -1, \\ \frac{1}{2} \left| \frac{(1/R)^{\tilde{d}} - R^{\tilde{d}}}{(r/R)^{\tilde{d}} - (R/r)^{\tilde{d}}} \right|, & s_\chi = +1, \end{cases} \quad (268)$$

$$\rho(\vartheta) = \begin{cases} \frac{1}{4} \left| (1/R)^{\tilde{d}} + R^{\tilde{d}} \right| \left| \sin \left(\frac{4\tilde{d}\vartheta}{(1/R)^{\tilde{d}} + R^{\tilde{d}}} + 2 \arctan R^{\tilde{d}} \right) \right|, & s_\chi = -1, \\ \frac{1}{4} \left| (1/R)^{\tilde{d}} - R^{\tilde{d}} \right| \left| \sinh \left(\frac{4\tilde{d}\vartheta}{|(1/R)^{\tilde{d}} - R^{\tilde{d}}|} + 2 \text{Arth} R^{\tilde{d}} \right) \right|, & s_\chi = +1. \end{cases} \quad (269)$$

From the definition of function χ (251) we have:

$$e^{B(r)} = \exp\left(\frac{1}{\tilde{d}}\chi(r) - \sum_I \frac{d_I}{\tilde{d}}A_I(r)\right), \quad (270)$$

and:

$$e^{B_\vartheta(\vartheta)} = \exp\left(-\sum_I \frac{d_I}{\tilde{d}}A_I(\vartheta) + \frac{\epsilon_\chi}{\tilde{d}}\right) \frac{\rho(\vartheta)^{-(1+\frac{1}{\tilde{d}})}}{2}, \quad (271)$$

Interesting, that if one substitutes (265) into (245) one then obtains:

$$B_\vartheta'' + B_\vartheta' \left(\chi' + \frac{\tilde{d}+1}{r}\right) - \frac{\tilde{d}(\tilde{d}+1)}{r^2} = -\frac{\sum_i D_i(S'_i)^2}{2(D-2)}. \quad (272)$$

The left hand side of the equation is similar to (257) but in addition the $\tilde{d}(\tilde{d}+1)/r^2$ component appears. The component is the origin for the $\rho^{-(1+1/\tilde{d})}$ factor in (271).

Thanks to (270, 251) and the already known function χ we are able to reduce a number of unknown functions in the equations of motion i.e. to replace B (or B_ϑ) by certain combination of A_I 's and to drop the equation (245) which does not contain any independent piece of information. The remaining equations (244–248) can be translated to a system depending on the ϑ variable:

$$\ddot{A}_I = \frac{\sum_{i \in I} \tilde{D}_i(\dot{S}_i)^2 - \sum_{i \notin I} D_i(\dot{S}_i)^2}{2(D-2)}, \quad (273)$$

$$\ddot{\phi}_\alpha = -\frac{1}{2} \sum_i \varsigma_i a_{i\alpha} (\dot{S}_i)^2, \quad (274)$$

$$\ddot{S}_i = -\left(\frac{1}{2} \sum_\alpha a_{i\alpha} \dot{\phi}_\alpha - \sum_{I:i \in I} d_I \dot{A}_I\right) \dot{S}_i, \text{ (electric)} \quad (275)$$

$$\frac{1}{\tilde{d}} \left(\sum_I d_I \dot{A}_I \right)^2 + \sum_I (d_I(\dot{A}_I)^2) + \frac{1}{2} \sum_\alpha (\dot{\phi}_\alpha)^2 + \Lambda_\chi = \frac{1}{2} \sum_i (\dot{S}_i)^2, \quad (276)$$

where:

$$\dot{S}_i = \begin{cases} \sigma_i(e^{C_i}) \exp\left(\frac{1}{2} \sum_\alpha a_{i\alpha} \phi_\alpha - \sum_{I:i \in I} d_I A_I\right), & \text{(electric)}, \\ -2\lambda_i e^{-\epsilon_\chi} \exp\left(\frac{1}{2} \sum_\alpha a_{i\alpha} \phi_\alpha + \sum_{I:i \in I} d_I A_I\right), & \text{(magnetic)}, \end{cases} \quad (277)$$

$$\Lambda_\chi = -\frac{16\tilde{d}(\tilde{d}+1)c_\chi}{(c_\chi - 1)^2}. \quad (278)$$

The system (273–278) together with (269) and (271) carries complete information originally contained in (244–248, 250). Since the r -variable system (244–248, 250) drastically simplifies when harmonic gauge is imposed, it is interesting what happens to (273 – 278, 271, 269) in an analogous situation. If one treats the ϑ as a fundamental coordinate then all dependence of the solution on the parameter R (so also all differences between the harmonic and a general cases) enters only in two places: in the ρ function (269) which influences a form of the B_ϑ function (271) and in the Λ_χ constant (278) appearing in (276). However (276) is not a dynamic equation but rather a constraint decreasing by one a number of integration constants.

5.1.5 The Δ matrix and reduction to Toda-like system.

Define:

$$\omega_i = \exp\left(\frac{1}{2} \sum_\alpha \varsigma_i a_{i\alpha} \phi_\alpha - \sum_{I:i \in I} d_I A_I\right). \quad (279)$$

With these functions one can find that (275) leads to:

$$\dot{S}_i = p_i/\omega_i, \quad (280)$$

where p_i are nonzero real integration constants. Simultaneously (277) for magnetic branes gives:

$$\dot{S}_i = \frac{-2\lambda_i e^{-\epsilon_x}}{\omega_i}, \quad (281)$$

so, after the identification:

$$p_i = -2\lambda_i e^{-\epsilon_x}, \quad \sigma_i = \text{sgn}\lambda_i, \quad (282)$$

the relation (280) is valid for electric as well as for magnetic branes.

It can be checked from (273–274) that ω_i have to satisfy the following system of equations:

$$\frac{d^2}{d\vartheta^2}(\ln |\omega_i|) = -\sum_j \Delta_{ij} \frac{p_j^2}{4\omega_j^2}, \quad (283)$$

what is equivalent to a Toda-like system. A proof of the equivalence, together with a short introduction to a theory of Toda systems and methods of solving such kind of differential equations will be given in the section 5.2.

Elements of Δ matrix are:

$$\Delta_{ij} = \frac{2}{D-2} \left(\sum_{\bar{I}: i, j \in \bar{I}} d_{\bar{I}} \sum_{\bar{J}: i, j \notin \bar{J}} d_{\bar{J}} - \sum_{\bar{I}: i \in \bar{I}, j \notin \bar{I}} d_{\bar{I}} \sum_{\bar{J}: i \notin \bar{J}, j \in \bar{J}} d_{\bar{J}} \right) + \sum_{\alpha} \varsigma_i a_{i\alpha} \varsigma_j a_{j\alpha}, \quad (284)$$

where indices \bar{I}, \bar{J} run through all values allowed for I, J and additionally \emptyset , and by d_{\emptyset} is understood \tilde{d} (but not $\dim V_{\emptyset}$). By the definition (284) the matrix Δ is symmetric. Its diagonal elements obey:

$$\Delta_{ii} = \frac{2D_i \tilde{D}_i}{D-2} + \sum_{\alpha} a_{i\alpha}^2 \quad (285)$$

and the non-diagonal ones are bounded by:

$$\Delta_{ij} \leq \frac{1}{2}(\Delta_{ii} + \Delta_{jj}). \quad (286)$$

If $\det(\Delta) \neq 0$ then it is possible to express all functions A_I, B_{ϑ} (264), ϕ_{α} (221), C_i (223) in terms of ω_i :

$$\exp(A_I(\vartheta)) = E_I \left(\prod_i \omega_i(\vartheta)^{\gamma_I^i} \right) \exp(c_I \vartheta), \quad (287)$$

$$\exp(B_{\vartheta}(\vartheta)) = E_B \left(\prod_i \omega_i(\vartheta)^{\gamma_B^i} \right) \exp(c_B \vartheta) \rho(\vartheta)^{-(1+1/\tilde{d})}, \quad (288)$$

$$\exp(\phi_{\alpha}(\vartheta)) = E_{\alpha} \left(\prod_i \omega_i(\vartheta)^{\gamma_{\alpha}^i} \right) \exp(c_{\alpha} \vartheta), \quad (289)$$

$$\frac{d}{d\vartheta} \exp(C_i(\vartheta)) = \frac{p_i \sigma_i}{\omega_i(\vartheta)^2}, \quad (290)$$

where γ_I^i , γ_B^i and γ_{α}^i have to satisfy:

$$\frac{D-2}{2} \sum_i \Delta_{ij} \gamma_I^i = \begin{cases} -\tilde{D}_j & \text{if } j \in I, \\ D_j & \text{if } j \notin I, \end{cases} \quad (291)$$

$$\sum_i \Delta_{ij} \gamma_{\alpha}^i = 2a_{j\alpha}, \quad (292)$$

$$\gamma_B^i = -\frac{1}{\tilde{d}} \sum_I d_I \gamma_I^i. \quad (293)$$

It can be checked that if all $a_{i\alpha}$ vanish, the numbers γ_I^i and γ_B^i are in an excellent agreement with the harmonic function rule presented in (202 – 204). Values of real constants c_I , c_B , c_α and positive constants E_I , E_B , E_α are restricted by:

$$0 = \frac{1}{2} \sum_{\alpha} \varsigma_i a_{i\alpha} c_\alpha - \sum_{I:i \in I} d_I c_I, \quad (294)$$

$$\prod_{I:i \in I} E_I^{d_I} = \prod_{\alpha} E_\alpha^{\frac{1}{2} \varsigma_i a_{i\alpha}}, \quad (295)$$

$$0 = \sum_I d_I c_I + \tilde{d} c_B, \quad (296)$$

$$\frac{e^{\epsilon \chi}}{2} = \left(\prod_I E_I^{d_I} \right) E_B^{\tilde{d}}. \quad (297)$$

So the problem of finding brane solution in gravity coupled to an arbitrary number of antisymmetric tensors and scalar fields without assumption of harmonic gauge can be reduced to solving the Toda-like system (283) with a constraint derived from (276):

$$\sum_{ij} (\Delta^{-1})_{ij} \frac{\dot{\omega}_i \dot{\omega}_j}{\omega_i \omega_j} + \frac{1}{2} (\Lambda_\chi + \Lambda_c) = \sum_i \frac{p_i^2}{4\omega_i^2}, \quad (298)$$

where:

$$\Lambda_c = \sum_I d_I c_I^2 + \tilde{d} c_B^2 + \frac{1}{2} \sum_{\alpha} c_{\alpha}^2. \quad (299)$$

Note, that if the model describes a single brane without dilaton the solution is constructed only of e^{B_θ} and one e^{A_I} , the constraints (294) and (296) enforce the constants c_I and c_B to vanish. Consequently the identity $\Lambda_c = 0$ holds.

5.1.6 Solution for $\tilde{d} = 0$.

Defining the function χ (251) we imposed to the discussed model additional constraint that $\tilde{d} \neq 0$ what means that the overall transverse space cannot be two dimensional. Now let us turn our attention to the opposite case, i.e. we assume now that $\tilde{d} = 0$. In such situation the equations of motion expressed in terms of the scalar functions A_I, B, C_i, ϕ_α (244 – 248) simplify to:

$$A''_I + A'_I \left(\sum_J d_J A'_J + \frac{1}{r} \right) = \frac{\sum_{i \in I} \tilde{D}_i (S'_i)^2 - \sum_{i \notin I} D_i (S'_i)^2}{2(D-2)}, \quad (300)$$

$$B'' + B' \left(\sum_I d_I A'_I + \frac{1}{r} \right) + \frac{1}{r} \sum_I d_I A'_I = -\frac{\sum_i D_i (S'_i)^2}{2(D-2)}, \quad (301)$$

$$\phi''_\alpha + \phi'_\alpha \left(\sum_I d_I A'_I + \frac{1}{r} \right) = -\frac{1}{2} \sum_i \varsigma_i a_{i\alpha} (S'_i)^2, \quad (302)$$

$$\sum_I d_I \left(A''_I - \frac{1}{r} A'_I - 2 A'_I B' + (A'_I)^2 \right) = \frac{1}{2} \sum_i (S'_i)^2 - \frac{1}{2} \sum_{\alpha} (\phi'_\alpha)^2, \quad (303)$$

$$C'_i \left(C'_i - \sum_{I:i \in I} d_I A'_I + \sum_{I:i \notin I} d_I A'_I + \sum_{\alpha} a_{i\alpha} \phi'_\alpha + \frac{1}{r} \right) = -C''_i, \quad (304)$$

where:

$$\varsigma_i = \begin{cases} +1 & (\text{electric}), \\ -1 & (\text{magnetic}), \end{cases} \quad (305)$$

$$S'_i = \begin{cases} \sigma_i(e^{C_i})' \exp\left(\frac{1}{2} \sum_{\alpha} a_{i\alpha} \phi_{\alpha} - \sum_{I:i \in I} d_I A_I\right) & (\text{electric}), \\ \frac{\lambda_i}{r^{d+1}} \exp\left(\frac{1}{2} \sum_{\alpha} a_{i\alpha} \phi_{\alpha} - \sum_{I:i \notin I} d_I A_I\right) & (\text{magnetic}). \end{cases} \quad (306)$$

Furthermore the definition of the χ function (251) gives:

$$\chi = \sum_I d_I A_I \quad (307)$$

and the function has to satisfy:

$$\chi'' + (\chi')^2 + \frac{1}{r} \chi' = 0. \quad (308)$$

Solving the above equation one immediately obtains:

$$\chi(r) = \ln \left| \frac{c_{\chi} - \ln(1/r)}{c_{\chi} - c_0} \right| + \epsilon_{\chi}(c_0). \quad (309)$$

We see that the factor $1/r^{2\tilde{d}}$ appearing in (253) is replaced here by $\ln(1/r)$. The parameter c_{χ} takes not only real but also infinite values and the limits of χ for $c_{\chi} = \pm\infty$ are equal:

$$\lim_{c_{\chi} \rightarrow \pm\infty} \chi(r; c_{\chi}) = \epsilon_{\chi}, \quad (310)$$

so we can identify these points in the space of parameters analogously as it was done for the $\tilde{d} \neq 0$ case. We can also introduce the parameter R , which in this situation instead of (255) has to obey:

$$c_{\chi} = \ln \left(\frac{1}{R} \right), \quad (311)$$

what after setting $c_0 = 0$ allows us to rewrite (309) as:

$$\chi(r) = \ln \left| \frac{\ln(R/r)}{\ln R} \right| + \epsilon_{\chi}. \quad (312)$$

Note that in distinction to (255) it is not necessary to introduce discrete parameter s_{χ} because for positive R the logarithm $\ln R$ takes positive as well as negative values.

The equation for ϑ is now:

$$\vartheta'' + \chi' \vartheta' + \frac{1}{r} \vartheta' = 0. \quad (313)$$

so it is satisfied in general by $\vartheta = a\chi + b$ for arbitrary real constants a and b which again can be independently chosen for areas where $r > R$ and $r < R$ respectively. Take then:

$$\vartheta(r) = \begin{cases} -|\ln R| \ln \left| \frac{\ln(R/r)}{\ln R} \right| & \text{for, } r < L, \\ |\ln R| \ln \left| \frac{\ln(R/r)}{\ln R} \right| & \text{for, } r > L, \end{cases} \quad (314)$$

what preserves a possibility of the identification $c_{\chi} = +\infty$ ($R = 0$) with $c_{\chi} = -\infty$ ($R = \infty$) because gives:

$$\lim_{R \rightarrow 0} \vartheta(r; R) = \lim_{R \rightarrow \infty} \vartheta(r; R) = \ln \frac{1}{r}. \quad (315)$$

If $R \neq 1$ it is possible to make a coordinate change which replaces r with ϑ . But in distinction to the situation of $\tilde{d} \neq 0$, in this case r usually covers wider area of spacetime than ϑ . More precisely for $R \in (0, \infty)$, when r ranges from 0 to R , the coordinate ϑ runs over all real values and makes it one again when $r \in (R, \infty)$. So the change of the coordinates is well defined only if we restrict the discussion to one of the areas. But if $R = 0$ or $R = \infty$ then from (315) we have $\vartheta = -\ln r$ what works properly for all positive r .

The spacetime interval in the coordinates ϑ is the same as (264) and ρ and B_ϑ are still defined by (266) and (265) respectively, so:

$$\rho = \exp\left(-\frac{\vartheta}{|\ln R|}\right) = \left|\frac{\ln R}{\ln(R/r)}\right|. \quad (316)$$

But for B_ϑ we are not able to write an analog of (271) because this function does not contribute to χ if $\tilde{d} = 0$. The equations of motion in terms of ϑ are then:

$$\ddot{A}_I = \frac{\sum_{i \in I} \tilde{D}_i (\dot{S}_i)^2 - \sum_{i \notin I} D_i (\dot{S}_i)^2}{2(D-2)}, \quad (317)$$

$$\ddot{\phi}_\alpha = -\frac{1}{2} \sum_i \varsigma_i a_{i\alpha} (\dot{S}_i)^2, \quad (318)$$

$$\ddot{B}_\vartheta = -\frac{\sum_i D_i (\dot{S}_i)^2}{2(D-2)}, \quad (319)$$

$$\ddot{S}_i = -\left(\frac{1}{2} \sum_\alpha a_{i\alpha} \dot{\phi}_\alpha - \sum_{I:i \in I} d_I \dot{A}_I\right) \dot{S}_i, \quad (\text{electric}), \quad (320)$$

$$\frac{1}{c_\chi^2} - \frac{2\text{sgn}(r-R)}{c_\chi} \dot{B}_\vartheta + \sum_I \left(d_I (\dot{A}_I)^2\right) + \frac{1}{2} \sum_\alpha (\dot{\phi}_\alpha)^2 = \frac{1}{2} \sum_i (\dot{S}_i)^2. \quad (321)$$

The system can be solved analogously as in the case of $\tilde{d} \neq 0$ by introducing the functions ω_i (279). But because for $\tilde{d} = 0$ the Toda-like system (283) does not carry information necessary to determine a form of the B_ϑ we have to solve additionally (319). So as a result of solving the Toda-like system, if $\det \Delta \neq 0$ we obtain for the functions A_I and ϕ_α :

$$\exp(A_I(\vartheta)) = E_I \left(\prod_i \omega_i(\vartheta)^{\gamma_I^i} \right) \exp(c_I \vartheta), \quad (322)$$

$$\exp(\phi_\alpha(\vartheta)) = E_\alpha \left(\prod_i \omega_i(\vartheta)^{\gamma_\alpha^i} \right) \exp(c_\alpha \vartheta), \quad (323)$$

where γ_I^i and γ_α^i have to satisfy:

$$\frac{D-2}{2} \sum_i \Delta_{ij} \gamma_I^i = \begin{cases} -\tilde{D}_j & \text{if } j \in I, \\ D_j & \text{if } j \notin I, \end{cases} \quad (324)$$

$$\sum_i \Delta_{ij} \gamma_\alpha^i = 2a_{j\alpha} \quad (325)$$

and values of real constants c_I , c_α and positive constants E_I , E_α are restricted by:

$$0 = \frac{1}{2} \sum_\alpha \varsigma_i a_{i\alpha} c_\alpha - \sum_{I:i \in I} d_I c_I, \quad (326)$$

$$\prod_{I:i \in I} E_I^{d_I} = \prod_\alpha E_\alpha^{\frac{1}{2} \varsigma_i a_{i\alpha}}, \quad (327)$$

$$0 = \sum_I d_I c_I, \quad (328)$$

$$\frac{e^{\epsilon_\chi}}{2} = \left(\prod_I E_I^{d_I} \right). \quad (329)$$

But for the remaining functions C_i and B_ϑ we have to solve the following equations:

$$\frac{d^2}{d\vartheta^2} \ln B_\vartheta(\vartheta) = -\frac{1}{2(D-2)} \sum_i \frac{D_i p_i^2}{\omega_i(\vartheta)^2}, \quad (330)$$

$$\frac{d}{d\vartheta} \exp(C_i(\vartheta)) = \frac{p_i \sigma_i}{\omega_i(\vartheta)^2} \quad (331)$$

and to remove one integration constant by use of the constraint (321).

Note also that combining the system (283) with the equation (319) we again obtain a Toda-like system:

$$\frac{d^2}{d\vartheta^2} \ln \omega_{\bar{i}} = - \sum_{\bar{j}} \bar{\Delta}_{\bar{i}\bar{j}} \frac{p_{\bar{j}}^2}{4\omega_{\bar{j}}^2}, \quad (332)$$

where the indices \bar{i}, \bar{j} run through $1, \dots, N_A$ as i, j but additionally can take value 0 and:

$$p_0 = 0, \quad (333)$$

$$\omega_0 = \ln B_\vartheta, \quad (334)$$

$$\bar{\Delta}_{\bar{i}\bar{j}} = \begin{pmatrix} \Delta_{00}, & \Delta_{0j} \\ \Delta_{i0}, & \Delta_{ij} \end{pmatrix}, \quad (335)$$

$$\Delta_{00} = 0, \quad (336)$$

$$\Delta_{0j} = \frac{2D_j}{D-2}, \quad (337)$$

$$\Delta_{i0} = 0. \quad (338)$$

5.1.7 Charges and masses

Each of the branes in the system under consideration carry some Ramond-Ramond charge. For the magnetic branes the charge is defined by (78) what immediately gives:

$$Q_m^i = \int_{\hat{V}_i \times \partial V_\emptyset} F^i = \Omega_{(\tilde{d}+1)} |\hat{V}_i| \lambda_i. \quad (339)$$

For the elementary branes we have to count the electric charge from (75) and obtain:

$$Q_e^i = \int_{\hat{V}_i \times \partial V_\emptyset} \exp \left(\sum_\alpha a_{i\alpha} \phi_\alpha \right) * F^i = \Omega_{(\tilde{d}+1)} |\hat{V}_i| q_e^i, \quad (340)$$

where the quantity q_e is defined by a formula which seems to be very complicated:

$$q_e^i = \sigma_i \exp \left(\sum_\alpha a_{i\alpha} \phi_\alpha - \sum_{I:i \in I} d_I A_I + \sum_{I:i \notin I} d_I A_I + \tilde{d} B \right) (e^{C_i})' r^{\tilde{d}+1}. \quad (341)$$

but after more detailed analysis proves to be a constant number:

$$q_e^i = \lambda_i. \quad (342)$$

We can also calculate energy of the intersecting branes system. Let us first see that for the gravitational tensors like Ricci tensor, Ricci scalar and Einstein tensor:

$$G_{MN} = R_{MN} - \frac{1}{2}g_{MN}R, \quad (343)$$

thanks to the equations of motion (244 – 248) is it possible to write them in terms of the functions S'_i (250):

$$R_{\mu^I \nu^I} = -e^{2(A_I - B)} \frac{\sum_{i \in I} \tilde{D}_i(S'_i)^2 - \sum_{i \notin I} D_i(S'_i)^2}{2(D-2)} \eta_{\mu^I \nu^I}, \quad (344)$$

$$R_{mn} = \frac{\sum_i D_i(S'_i)^2}{2(D-2)} \delta_{mn} - \frac{1}{2} \sum_i (S'_i)^2 \frac{y_m y_n}{r^2} + \frac{1}{2} \sum_\alpha (\phi'_\alpha)^2 \frac{y_m y_n}{r^2}, \quad (345)$$

$$R = e^{-2B} \left(\frac{\sum_i (D_i - \tilde{D}_i)(S'_i)^2}{2(D-2)} + \frac{1}{2} \sum_\alpha (\phi'_\alpha)^2 \right), \quad (346)$$

$$G_{\mu^I \nu^I} = -\frac{1}{4} e^{2(A_I - B)} \left(\sum_{i \in I} (S'_i)^2 - \sum_{i \notin I} (S'_i)^2 + \sum_\alpha (\phi'_\alpha)^2 \right) \eta_{\mu^I \nu^I}, \quad (347)$$

$$G_{mn} = \left(\sum_I (S'_i)^2 - \sum_\alpha (\phi'_\alpha)^2 \right) \left(\frac{1}{4} \delta_{mn} - \frac{y_m y_n}{2r^2} \right), \quad (348)$$

So, the component with double timelike indices of the stress-energy tensor is:

$$T^t_t = G^t_t = e^{-2B} \left(\sum_i (S'_i)^2 + \sum_\alpha (\phi'_\alpha)^2 \right) \quad (349)$$

and an integral of it:

$$\mathcal{E} = \int_{\mathcal{M}} d^D X \sqrt{|\det g|} T^t_t \quad (350)$$

$$= \Omega_{(\tilde{d}+1)} |V| \int dr r^{\tilde{d}+1} e^\chi \left(\sum_i (S'_i)^2 + \sum_\alpha (\phi'_\alpha)^2 \right) \quad (351)$$

$$= -\frac{e^{\epsilon_\chi}}{2} \Omega_{(\tilde{d}+1)} |V| \int d\vartheta \left(\sum_i (\dot{S}_i)^2 + \sum_\alpha (\dot{\phi}_\alpha)^2 \right) \quad (352)$$

$$= -\frac{e^{\epsilon_\chi}}{2} \Omega_{(\tilde{d}+1)} |V| \int d\vartheta \left(\sum_i \left(\frac{p_i}{\omega_i} \right)^2 + \sum_\alpha \left(\sum_i \gamma_\alpha^i \frac{\dot{\omega}_i}{\omega_i} + c_\alpha \right)^2 \right). \quad (353)$$

The above integration should be taken over the whole space to cover the whole energy density which can contribute to the total energy. However it will be shown later (see the paragraph 5.4.2) that "the whole space" seen by a given observer is not necessary the same as the area described by $r \in (0, \infty)$ or $\vartheta \in (\vartheta(r=0), \vartheta(r=\infty))$ and to determine the area properly some additional analysis is needed.

5.2 Toda and Toda-like systems.

The Toda system is a system of second order differential equations which can be written in a form [128]:

$$\ddot{x}_i = - \sum_j K_{ij} \exp(-x_j). \quad (354)$$

where K_{ij} is a Cartan matrix i.e. a matrix satisfying:

$$K_{ii} = 2, \quad (355)$$

$$K_{ij} \leq 0, \quad \text{for } i \neq j, \quad (356)$$

$$K_{ij} = 0 \Leftrightarrow K_{ji} = 0. \quad (357)$$

After the substitution:

$$q_i = (K^{-1})_{ij} x_j, \quad (358)$$

the Toda system is reformulated as:

$$\ddot{q}_i = -\exp\left(-\sum_j K_{ij} q_j\right). \quad (359)$$

The above equations can be derived from a following lagrangian:

$$L = -\frac{1}{2} \sum_{ij} K_{ij} \dot{q}_i \dot{q}_j - \sum_i \exp\left(-\sum_j K_{ij} q_j\right). \quad (360)$$

This type of systems was studied in the context of gauge theories on lattices [129, 130], where the matrix K_{ij} was understood as a Cartan matrix of a simple Lie algebra corresponding to a given compact Lie group describing the considered gauge symmetry. There are four infinite series of Lie algebras $A(n)$, $B(n)$, $C(n)$ and $D(n)$ and five exceptional examples $E(6)$, $E(7)$, $E(8)$, $F(3)$ and $G(2)$. Let us review them and their Cartan matrices.

- $A(n)$ or $sl(n+1, \mathbf{C})$:

$$K_{A(n)} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}, \quad (361)$$

- $B(n)$ or $so(2n+1)$:

$$K_{B(n)} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -2 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}, \quad (362)$$

- $C(n)$ or $sp(n, \mathbf{C})$:

$$K_{C(n)} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -2 & 2 \end{pmatrix}, \quad (363)$$

- $D(n)$ or $so(2n)$:

$$K_{D(n)} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 0 & 2 \end{pmatrix}, \quad (364)$$

- $E(6)$:

$$K_{E(6)} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}, \quad (365)$$

- $E(7)$:

$$K_{E(7)} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 2 \end{pmatrix}, \quad (366)$$

- $E(8)$:

$$K_{E(8)} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \quad (367)$$

- $F(4)$:

$$K_{F(4)} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \quad (368)$$

- $G(2)$:

$$K_{G(2)} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}. \quad (369)$$

Turning back to the problem of intersecting branes we see that after the substitution:

$$x_i = -2 \ln\left(\frac{p_i \sqrt{\Delta_{ii}}}{2\omega_i}\right) \quad (370)$$

into (283) the equations take the form of (354) where:

$$K_{ij} = \frac{2\Delta_{ij}}{\Delta_{jj}}. \quad (371)$$

But for arbitrary Δ the matrix K does not have to obey the Cartan matrix conditions (355) – (357). Such generalizations of the Toda systems are called Toda-like systems.

The fact that the models of the intersecting branes in full generality reduce to Toda-like but not exactly Toda systems causes a problem since methods of integration are developed mainly for the latter ones. Fortunately quite large set of reasonable brane configurations can be described in terms of the standard Toda systems.

It is important to note that while any given matrix Δ is necessarily symmetric the corresponding matrix K defined by the identity (371) may be not symmetric. So those of the above algebras which have not symmetric Cartan matrices can also be relevant for some configurations of the intersecting branes. Another very useful fact which significantly enlarges set of known solvable cases is that if any matrix K can be written in a block diagonal form:

$$K = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix} \quad (372)$$

then the problem of solving the corresponding Toda system decomposes into two separate problems related with K_1 and K_2 respectively. On a ground of Lie algebras it means that from solutions of the simple Lie algebras one can immediately construct solutions with semisimple algebras being a sum of the simple ones. In particular the case of diagonal matrix K is related to semi-simple $\oplus^n A(1)$ algebra and can be treated as n independent equations. Each of the equations is the Liouville equation:

$$\ddot{q} = -e^{-2q}, \quad (373)$$

which can be solved:

$$e^{q(\vartheta)} = \begin{cases} \frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa}(\vartheta - \theta)), & \text{for } \kappa > 0, \\ (\vartheta - \theta), & \text{for } \kappa = 0, \\ \frac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa}(\vartheta - \theta)), & \text{for } \kappa < 0, \end{cases} \quad (374)$$

where θ is a real constant.

A very elegant method leading to a solution of the Toda system related to $A(n)$ algebra was given in [131] and furthermore extended to $B(n)$ and $C(n)$ and applied to the problem of intersecting branes under assumption of the harmonic gauge [124]. But of course the method works also in a general situation without the harmonic gauge.

The solution for $A(n)$ is then:

$$e^{-q_k(\vartheta)} = i^{k(n+1-k)} \sum_{j_1 < \dots < j_k}^{n+1} f_{j_1} \dots f_{j_k} V^2(j_1, \dots, j_k) e^{(\mu_{j_1} + \dots + \mu_{j_k})\vartheta}, \quad (375)$$

where V is the Vandermonde determinant defined as:

$$V(j_1, \dots, j_k) = \prod_{j_m < j_l} (\mu_{j_m} - \mu_{j_l}), \quad V(\mu_j) = 1, \quad (376)$$

the complex constants f_j , μ_j satisfy:

$$\prod_{j=1}^{n+1} f_j = V^{-2}(1, \dots, n+1), \quad \sum_{j=1}^{n+1} \mu_j = 0 \quad (377)$$

and additionally the constraint (298).

5.3 Diagonal Δ solution.

Consider the case when matrix Δ is diagonal and nonsingular. Then with use of (374) the equation (283) gives:

$$\omega_i(\vartheta) = \begin{cases} \left| \frac{p_i \sqrt{\Delta_{ii}}}{2\sqrt{\kappa_i}} \sin(\sqrt{\kappa_i}(\vartheta - \theta_i)) \right|, & \text{for } \kappa_i > 0, \\ \left| \frac{p_i \sqrt{\Delta_{ii}}}{2} (\vartheta - \theta_i) \right|, & \text{for } \kappa_i = 0, \\ \left| \frac{p_i \sqrt{\Delta_{ii}}}{2\sqrt{-\kappa_i}} \sinh(\sqrt{-\kappa_i}(\vartheta - \theta_i)) \right|, & \text{for } \kappa_i < 0, \end{cases} \quad (378)$$

where real phases θ_i are independent but (298) gives a restriction on κ_i which for $\tilde{d} \neq 0$ reads:

$$\sum_i \frac{\kappa_i}{\Delta_{ii}} = \frac{1}{2}(\Lambda_x + \Lambda_c). \quad (379)$$

Substituting (378) into (287–289) and solving (290):

$$e^{C_i(\vartheta)} = \begin{cases} E_i - \frac{4\sqrt{\kappa_i}}{p_i \Delta_{ii}} \cot(\sqrt{\kappa_i}(\vartheta - \theta_i)), & \text{for } \kappa_i > 0, \\ E_i - \frac{4}{p_i \Delta_{ii}} (\vartheta - \theta_i)^{-1}, & \text{for } \kappa_i = 0, \\ E_i - \frac{4\sqrt{-\kappa_i}}{p_i \Delta_{ii}} \coth(\sqrt{-\kappa_i}(\vartheta - \theta_i)), & \text{for } \kappa_i < 0, \end{cases} \quad (380)$$

where E_i are integration constants. Therefore in this case we have an explicit form of the solution.

It is not necessary to apply the absolute value to the ω_i in (378). However if any ω_i is a solution of (283) then $-\omega_i$ and more general any $\tilde{\omega}_i(\vartheta) = j(\vartheta)\omega_i(\vartheta)$ such that $|j| = 1$ is also a good solution of the system of equations everywhere except points where j is discontinuous. But because the equation (283) itself is not well defined for $\omega_i = 0$, what means $\vartheta = \theta$ there is no obstacle to assume that j can change sign there. It gives a possibility to write the functions ω_i as everywhere nonnegative which is in an agreement with the definition (279).

5.4 Single component solution.

Let us restrict our attention to the simplest case – the single component solution which was already discussed in the section 4.1 with the harmonic gauge condition imposed. Let us now consider the case when the condition is dropped.

5.4.1 Solution without dilaton for $\tilde{d} \geq 1$ – general remarks.

If $N_A = 1$, $N_\phi = 0$ and $\tilde{d} \neq 0$ the above solution can be written as:

$$e^A = \omega^{-\frac{1}{d}}, \quad (381)$$

$$e^{B_\vartheta} = \left(\frac{1}{2} e^{\epsilon_x} \right)^{\frac{1}{d}} \omega^{\frac{1}{d}} \rho^{-1-\frac{1}{d}}, \quad (382)$$

with $e^C = e^{C_{i=1}}$ given by (380), $\omega = \omega_{i=1}$ by (378) and $E_C = E_{i=1}$, $p = p_{i=1}$, $\theta = \theta_{i=1}$, $\kappa = \kappa_{i=1}$, $\Delta = \Delta_{11}$, where:

$$\kappa = \frac{\Delta}{2} \Lambda_x. \quad (383)$$

The parameter E_C has no physical meaning because it describes only freedom of shift of the antisymmetric potential: $e^C \rightarrow E_C + e^C$ and values of θ and ϵ_x can be fixed if we assume that at a given point ϑ_0 the functions A and B_ϑ (or B) take certain values.

For example if with $\vartheta \rightarrow \vartheta_0$ the spacetime tends to flat one, it is natural to set $A(\vartheta_0) = B(\vartheta_0) = 0$ what equivalently means:

$$\omega(\vartheta_0) = 1, \quad \chi(\vartheta_0) = 0. \quad (384)$$

In such a situation calculating the energy density of the brane from a formula derived form (353):

$$\mathcal{E} = -\frac{e^{\epsilon_x} p^2}{2} \Omega_{(\tilde{d}+1)} |V| \int \frac{d\vartheta}{\omega^2} \quad (385)$$

we should take a boundary of the integration exactly at ϑ_0 what gives [112]:

$$\mathcal{E} = \Omega_{\tilde{d}+1} |V| \frac{2}{\sqrt{\Delta}} \sqrt{\lambda^2 - \frac{1}{2} \Lambda_\chi}. \quad (386)$$

When the solution is supersymmetric ($R = 0$ and consequently $\Lambda_\chi = 0$) it reproduces properly the BPS equality between mass and charge densities (154). And if $0 < R < \infty$ and $s_\chi = +1$ it gives $\mathcal{E} > Q$ as it should for nonsupersymmetric solutions. There is an exception for the solution with $R = \infty$ (which evidently is not supersymmetric because when applied to the electric 2-brane solution in $D = 11$ it does not satisfy (165)) – we have then $\mathcal{E} = Q$. This leads to a contradiction with the rule that the BPS states are supersymmetric. Moreover, for $s_\chi = -1$ we get even $\mathcal{E} < Q$.

This is only an apparent paradox, because we should remember that the formula (386) is well defined only when certain conditions are satisfied. The conditions which assure that performing the integration of (385) we keep all energy inside the integrated area but do not encounter any naked singularity. Additionally it may happen that it is impossible to impose the boundary conditions (384). So the validity of the formula (386) can be limited to only some subspace of the whole parameter space. To describe the subspace we need to examine more precisely some aspects of a geometry of various variants of the solution and find where there are singularities, horizons and boundary points at infinity.

Invariants of the metric.

Let us calculate invariants constructed by coefficients of the metric tensor:

$$R_{MN}R^{MN} = \frac{1}{4(D-2)^2} \left(\tilde{d}^2(d+1) + d^2(\tilde{d}+1) \right) e^{-4B} S'^4, \quad (387)$$

$$R = \frac{d-\tilde{d}}{2(D-2)} e^{-2B} S'^2, \quad (388)$$

$$G_{MN}G^{MN} = \frac{D}{16} e^{-4B} S'^4 \quad (389)$$

and check where the quantities take infinite or zero values. Their poles indicate the points which can be "suspected" to be singularities and the zeros indicate the points where the geometry tends to be flat. Except the situation when $d = \tilde{d}$ and the Ricci scalar identically vanishes all the quantities are proportional to a positive power of the function $e^{-B} S'$, so they all have zeros and poles at the same points. For $e^{-B} S'$ we have:

$$e^{-B} S' = e^{-B_\vartheta} \dot{S} = \text{const} \left(\frac{\rho}{\omega} \right)^{1+1/\tilde{d}}. \quad (390)$$

Therefore to determine poles and zeros it is enough to study properties of the functions ω and ρ . It is interesting that if $\tilde{d} = -1$ then all the invariants are constant – this case should be considered separately and we assume that $\tilde{d} \geq 1$ from now on. We know that the zeros of ω are given by $\vartheta = \theta$, so the points are "suspected" to describe localization of the singularities. But to verify real nature of the points it is very helpful to examine behavior of test particles freely falling onto them.

The proper and the coordinate time.

Because of the spherical symmetry of the model it is enough to look at a test particle moving only in the radial direction i.e. consider only time t and radial coordinate ϑ . The equation for geodesic in coordinates X^M reads:

$$\frac{d^2}{d\tau^2}X^M + \Gamma_{NR}^M \frac{d}{d\tau}X^N \frac{d}{d\tau}X^R = 0, \quad (391)$$

where τ describes an affine parameter and the Christoffel symbols are:

$$\begin{aligned} \Gamma_{tt}^t &= 0, & \Gamma_{tt}^\vartheta &= \dot{A}e^{2(A-B_\vartheta)}, \\ \Gamma_{\vartheta t}^t &= \dot{A}, & \Gamma_{\vartheta t}^\vartheta &= 0, \\ \Gamma_{\vartheta\vartheta}^t &= 0, & \Gamma_{\vartheta\vartheta}^\vartheta &= \dot{B}_\vartheta. \end{aligned} \quad (392)$$

Therefore equations for the trajectory of the test particle read:

$$\frac{d^2}{d\tau^2}t + 2\dot{A}\frac{d}{d\tau}t \frac{d}{d\tau}\vartheta = 0, \quad (393)$$

$$\frac{d^2}{d\tau^2}\vartheta + \dot{B}_\vartheta \left(\frac{d}{d\tau}\vartheta \right)^2 + \dot{A}e^{2(A-B_\vartheta)} \left(\frac{d}{d\tau}t \right)^2 = 0. \quad (394)$$

The first equation leads to:

$$E = e^{2A} \frac{d}{d\tau}t = \text{const}, \quad (395)$$

where E has interpretation of test particle energy and then the second equation can be rewritten as:

$$\frac{d^2}{d\tau^2}r + \dot{B}_\vartheta \left(\frac{d}{d\tau}r \right)^2 + \dot{A}e^{-2(A+B_\vartheta)} E^2 = 0. \quad (396)$$

Solving this equation one finally obtains:

$$\frac{d}{d\tau}r = e^{-B_\vartheta} \sqrt{C + E^2 e^{-2A}}, \quad (397)$$

where C is another integration constant. For radially propagating test particle one has:

$$ds^2 = -e^{2A}dt^2 + e^{2B_\vartheta}d\vartheta^2 = C d\tau^2. \quad (398)$$

Since any massive particle must have $ds^2 < 0$ it is convenient to set in that case $C = -1$, what defines units of the proper time. For photons and other massless particles we must set $C = 0$.

We should calculate the coordinate time δt and proper time (or affine parameter if the particle is massless) $\delta\tau$ that passes when the test particle falls from an arbitrary point ϑ_1 to the examined one ϑ_0 . For our needs it is enough to check if the values are finite or infinite for ϑ_1 arbitrarily close to ϑ_0 . These times, for the massive test particle, are:

$$\delta\tau = \int_{\vartheta_1}^{\vartheta_0} \frac{e^{B_\vartheta} d\vartheta}{\sqrt{E^2 e^{-2A} - 1}}, \quad (399)$$

$$\delta t = \int_{\vartheta_1}^{\vartheta_0} \frac{E e^{B_\vartheta - 2A} d\vartheta}{\sqrt{E^2 e^{-2A} - 1}} \quad (400)$$

and for the massless test particle:

$$\delta\tau = \int_{\vartheta_1}^{\vartheta_0} \frac{1}{E} e^{A+B_\vartheta} d\vartheta, \quad (401)$$

$$\delta t = \int_{\vartheta_1}^{\vartheta_0} e^{B_\vartheta - A} d\vartheta. \quad (402)$$

For the discussed model the last integrals can be rewritten as:

$$\delta\tau = \int_{\vartheta_1}^{\vartheta_0} \frac{1}{E} \omega^{-\frac{1}{d} + \frac{1}{d}} \rho^{-1 - \frac{1}{d}} d\vartheta, \quad (403)$$

$$\delta t = \int_{\vartheta_1}^{\vartheta_0} \omega^{\frac{1}{d} + \frac{1}{d}} \rho^{-1 - \frac{1}{d}} d\vartheta, \quad (404)$$

so everything depends on the functions ω and ρ .

5.4.2 Solution without dilaton for $\tilde{d} \geq 1$ – variants review.

We decompose the general solution into several variants with respect to values taken by parameters R , s_χ and θ .

We distinguish:

- Variant I where $R = 0$, what immediately gives $\kappa = 0$,
- Variant II where $R \in (0, \infty)$ and $s_\chi = +1$, so $\kappa < 0$,
- Variant III where $R = \infty$, so $\kappa = 0$,
- Variant IV where $R \in (0, \infty)$ and $s_\chi = -1$, so $\kappa > 0$

and

- Variant A when $\vartheta = \theta$ does not belong to an area described by positive r ,
- Variant B when $\vartheta = \theta$ is on an edge of the area with positive r ,
- Variant C when $\vartheta = \theta$ belongs to the area of positive r .

Variant IA where $R = 0$, $\kappa = 0$ and $\theta < -\frac{1}{2d}$. Then:

$$\rho = \left| \tilde{d}\vartheta + \frac{1}{2} \right|, \quad (405)$$

$$\omega = \frac{|p|\sqrt{\Delta}}{2} |\vartheta - \theta|, \quad (406)$$

ϑ	$-\infty$	θ	$-\frac{1}{2d}$	0	$+\infty$
r	undefined	undefined	∞	1	0
$\dot{S}e^{-B_\vartheta}$	finite	∞	0	finite	finite
δt	∞	finite	∞	finite	∞
$\delta\tau$	finite	finite	∞	finite	finite
e^A	0	∞	finite	finite	0
e^{B_ϑ}	0	0	∞	finite	0
e^B	undefined	undefined	finite	finite	∞

Table 2: Variant IA.

This gives exactly the supersymmetric single brane solution with a naked singularity hidden behind a horizon which was discussed in the section 4.1. The point $\vartheta = -\frac{1}{2d}$ (or equivalently $r = \infty$) describes

boundary of the spacetime at infinity where the geometry tends to the flat one. Because e^A and e^B are finite there it is possible to normalize them to one with the conditions (384) and get:

$$\left| \theta + \frac{1}{2\tilde{d}} \right| = \frac{2}{|p|\sqrt{\Delta}}, \quad (407)$$

$$\epsilon_\chi = 0. \quad (408)$$

The area where $\vartheta > -\frac{1}{2\tilde{d}}$ is related to positive values of the isotropic radial coordinate r and at $\vartheta = \infty$ (or $r = 0$) where metric tensor coefficients e^A and e^{B_ϑ} become singular but the invariants (387 – 389) are finite the horizon is located. The solution can be extended beyond the horizon by an identification of $\vartheta = +\infty$ with $\vartheta = -\infty$ and continued to $\vartheta = \theta$ where the naked singularity is settled. The singularity means a point which is reachable with finite $\delta\tau$ and where the invariants of the metric grows to infinity. An observer located at positive r can see only a region of the spacetime described by $\vartheta \in \left(-\frac{1}{2\tilde{d}}, \infty\right)$ (or equivalently $r \in (\infty, 0)$), because any piece of information sent to the observer from other points needs to travel infinite amount of coordinate time. So counting the total energy of the brane we have to integrate (385) over the area and get:

$$\mathcal{E} = -\frac{e^{\epsilon_\chi} p^2}{2} \Omega_{(\tilde{d}+1)} |V| \int_{\vartheta=\infty}^{\vartheta=-\frac{1}{2\tilde{d}}} \frac{d\vartheta}{\omega^2(\vartheta)} = \Omega_{(\tilde{d}+1)} |V| \frac{2}{\sqrt{\Delta}} |\lambda|. \quad (409)$$

Variant IB where $R = 0$, $\kappa = 0$ and $\theta = -\frac{1}{2\tilde{d}}$, so:

$$\rho = \left| \vartheta + \frac{1}{2\tilde{d}} \right|, \quad (410)$$

$$\omega = \frac{|p|\sqrt{\Delta}}{2} \left| \vartheta + \frac{1}{2\tilde{d}} \right|. \quad (411)$$

ϑ	$-\infty$	$\theta = -\frac{1}{2\tilde{d}}$	0	$+\infty$
r	undefined	∞	1	0
Se^{-B_ϑ}	finite	finite	finite	finite
δt	∞	finite	finite	∞
$\delta\tau$	finite	∞	finite	finite
e^A	0	∞	finite	0
e^{B_ϑ}	0	∞	finite	0
e^B	undefined	0	finite	∞

Table 3: Variant IB.

Then $\dot{S}e^{-B_\vartheta}$ is everywhere constant, non zero and finite what means that there is no singularity nor flat places at all. The point $\vartheta = -\frac{1}{2\tilde{d}}$ is characterized by $\delta\tau = \infty$, so we can interpret it as the boundary at infinity. But a geometry at the infinity is not flat and the metric tensor coefficients are singular so one should expect an external energy source attached at the point sustaining the nonzero curvature – the brane. Note also that a travel to the brane takes a finite amount of coordinate time t , so an observer located at positive r can "feel" a presence of the object. Integrating (385) from $\vartheta = \infty$ where a horizon is localised to $\vartheta = -\frac{1}{2\tilde{d}}$ and calculating in this way the total energy of the brane one gets an infinite value. It suggests that to evaluate the energy properly one should take into account also an external contribution originating from the point $\vartheta = \theta$. The solution can be continued beyond the horizon at $\vartheta = +\infty$ when it is identified with $\vartheta = -\infty$. At $\vartheta = -\frac{1}{2\tilde{d}}$ reached now from the left side one again encounters the external energy source. This situation (and all other type B variants)

are quite similar to a solution describing spacetime enclosed between two domain walls. But in this case we have not one but three or more direction transversal to the branes.

Variant IC where $R = 0$, $\kappa = 0$ and $\theta > -\frac{1}{2d}$. It is very similar to the variant IA but now the singularity appears in the area corresponding to positive r .

Variant IIC/A where $R \in (0, \infty)$, $s_\chi = +1$, $\kappa = \frac{\Delta}{2} \Lambda_\chi < 0$ and $\theta < \vartheta_0 = -\frac{1}{2d} \left| (1/R)^{\tilde{d}} - R^{\tilde{d}} \right| \text{Arth} R^{\tilde{d}}$ then:

$$\rho = \frac{1}{4} \left| (1/R)^{\tilde{d}} - R^{\tilde{d}} \right| \left| \sinh \left(\frac{4\tilde{d}\vartheta}{\left| (1/R)^{\tilde{d}} - R^{\tilde{d}} \right|} + 2\text{Arth} R^{\tilde{d}} \right) \right|, \quad (412)$$

$$\omega = \frac{p\sqrt{\Delta}}{2\sqrt{-\kappa}} \left| \sinh(\sqrt{-\kappa}(\vartheta - \theta)) \right|. \quad (413)$$

ϑ	$-\infty$	θ	ϑ_0	0	$+\infty$
r	R	finite	0∞	1	R
$\dot{S}e^{-B_\vartheta}$	0	∞	0	finite	0
δt	∞	finite	∞	finite	∞
$\delta\tau$	finite	finite	∞	finite	finite
e^A	0	∞	finite	finite	0
e^{B_ϑ}	0	0	∞	finite	0
e^B	∞	0	∞finite	finite	∞

Table 4: Variant IIC/A.

In the variant both coordinates $\vartheta \in [-\infty, +\infty]$ and $r \in [0, \infty]$ cover the same area of the spacetime. When $\vartheta \rightarrow \vartheta_0$ (or equivalently $r \rightarrow \infty$) a geometry of the spacetime tends to Minkowski spacetime and we can normalize e^A and e^B with (384). Going in the opposite direction at $\vartheta = \infty$ ($r = R$) one encounters a horizon. So we can give now a physical interpretation to the parameter R as a length where in the isotropic coordinate system the horizon is situated. When $R \rightarrow 0$ the localization goes to $r = 0$ and in this limit we recover the supersymmetric solution described by the variant IA. But let us turn back to the variant IIC/A. Near the horizon the quantity $\dot{S}e^{-B_\vartheta}$ behaves as:

$$\dot{S}e^{-B_\vartheta} = \exp \left(\frac{4(\tilde{d}+1) \left(1 - \sqrt{\frac{d\tilde{d}+d}{d+\tilde{d}}} \right)}{\left| 1/R^{\tilde{d}} - R^{\tilde{d}} \right|} \vartheta \right) \xrightarrow{\vartheta \rightarrow \infty} 0, \quad (414)$$

so it vanishes at $r = R$. Behind the horizon we can continue the solution with negative ϑ ($r < R$) and at $\vartheta = \theta$ we find a naked singularity. Since the singularity is placed in the area covered by r we should classify the variant as type C. But if we restrict our consideration to only $r > R$ we get a situation of type A, and this is an explanation why the variant is called IIC/A.

An observer living in the part of the spacetime where there is no singularity, it means at $r \in (R, \infty)$ sees only the area between the flat infinity at $r = \infty$ and the horizon $r = R$. So counting the energy density from (385) one should take exactly such limits of the integration. And this gives:

$$\mathcal{E} = \Omega_{\tilde{d}+1} |V| \frac{2}{\sqrt{\Delta}} \sqrt{\lambda^2 + \frac{8\tilde{d}(\tilde{d}+1)}{\left| 1/R^{\tilde{d}} - R^{\tilde{d}} \right|^2}}, \quad (415)$$

where we used the definition (278) of Λ_χ with $s_\chi = +1$. So the energy is always bigger than the charge of the brane as it should be expected for a nonsupersymmetric solution.

Variant IIB where $R \in (0, \infty)$, $s_\chi = +1$, $\kappa < 0$ and $\theta = -\frac{1}{2d} \left| (1/R)^{\tilde{d}} - R^{\tilde{d}} \right| \text{Arth}R^{\tilde{d}}$, so:

$$\rho = \frac{1}{4} \left| (1/R)^{\tilde{d}} - R^{\tilde{d}} \right| \left| \sinh \left(\frac{4\tilde{d}\vartheta}{\left| (1/R)^{\tilde{d}} - R^{\tilde{d}} \right|} + 2\text{Arth}R^{\tilde{d}} \right) \right|, \quad (416)$$

$$\omega = \frac{p\sqrt{\Delta}}{2\sqrt{-\kappa}} \left| \sinh \left(\sqrt{-\kappa} \left(\vartheta + \frac{1}{2\tilde{d}} \left| (1/R)^{\tilde{d}} - R^{\tilde{d}} \right| \text{Arth}R^{\tilde{d}} \right) \right) \right|. \quad (417)$$

ϑ	$-\infty$	θ	0	$+\infty$
r	R	$\infty 0$	1	R
$\dot{S}e^{-B_\vartheta}$	0	finite	finite	0
δt	∞	finite	finite	∞
$\delta\tau$	finite	∞	finite	finite
e^A	0	∞	finite	0
e^{B_ϑ}	0	∞	finite	0
e^B	∞	$\infty 0$	finite	∞

Table 5: Variant IIB.

The variant is quite similar to the IB, but now the isotropic coordinates cover the whole spacetime. Again we find the spacetime to be free of singularities and stretched between two points reachable in finite coordinate time where the curvature is finite but non zero, so we should expect the points to be localizations of the branes. Both the points are described in the ϑ coordinate system as $\vartheta = \theta$ (but as $r = 0$ and $r = \infty$) and they are separated by a horizon situated at $r = R$.

Variant IIA/C where $R \in (0, \infty)$, $s_\chi = +1$, $\kappa = \frac{\Delta}{2} \Lambda_\chi < 0$ and $\theta > \vartheta_0 = -\frac{1}{2d} \left| (1/R)^{\tilde{d}} - R^{\tilde{d}} \right| \text{Arth}R^{\tilde{d}}$.

It can be obtained from the variant IIC/A by shifting the point $\vartheta = \theta$ with the naked singularity to the area where $r > R$. So, the region "behind the horizon" described by $r < R$ is nonsingular now. We can count the energy density and find again (415). But to achieve it we should not determine ϵ_χ by the boundary conditions for e^B $r = \infty$, what is a point separated from the discussed area by the horizon and the singularity. We should rather make a coordinate change to introduce $\bar{r} = 1/r$ and normalize finite value of e^B to one at $\bar{r} = \infty$ what means $r = 0$. However, both the methods gives the same result $\epsilon_\chi = 0$.

Variant IIIA where $R = \infty$, $\kappa = 0$ and $\theta > \frac{1}{2d}$. Then the functions ρ and ω satisfy:

$$\rho = \left| \tilde{d}\vartheta - \frac{1}{2} \right|, \quad (418)$$

$$\omega = \frac{|p|\sqrt{\Delta}}{2} |\vartheta - \theta|. \quad (419)$$

While the variant IA can be understood as a some kind of a limit of the variant IIIC/A when R converges to 0, the variant IIIA is a similar limit of IIIA/C when $R \rightarrow \infty$. We have now a singularity at $\vartheta = \theta$ hidden behind a horizon located at $\vartheta = \pm\infty$ (or $r = \infty$). But for $\vartheta \in (-\infty, \frac{1}{2d})$ which corresponds to positive r the solution is regular. Finally at $\vartheta = \frac{1}{2d}$ ($r = 0$) a boundary at infinity is placed where the spacetime tends to the flat one. The last statement can be easily seen when the coordinate r is replaced by $\bar{r} = 1/r$. Then both functions e^A and e^B tend asymptotically to finite values with $\bar{r} \rightarrow \infty$. Taking the values as equal to 1 we get:

$$\left| \theta - \frac{1}{2\tilde{d}} \right| = \frac{2}{|p|\sqrt{\Delta}}, \quad (420)$$

$$\epsilon_\chi = 0, \quad (421)$$

ϑ	$-\infty$	0	$\frac{1}{2\tilde{d}}$	θ	$+\infty$
r	∞	1	0	undefined	undefined
$\dot{S}e^{-B_\vartheta}$	finite	finite	0	∞	finite
δt	∞	finite	∞	finite	∞
$\delta\tau$	finite	finite	∞	finite	finite
e^A	0	finite	finite	∞	0
e^{B_ϑ}	0	finite	∞	0	0
e^B	$O(r^{d/d-2})$	finite	∞	undefined	undefined

Table 6: Variant IIIA.

what furthermore gives:

$$\mathcal{E} = \Omega_{(\tilde{d}+1)} |V| \frac{2}{\sqrt{\Delta}} |\lambda|. \quad (422)$$

To achieve the positive sign in the above formula the integration leading to it has to be conducted from $\bar{r} = 0$ ($r = \infty$) to $\bar{r} = \infty$ ($r = 0$) instead of from $r = 0$ to $r = \infty$. But the obtained result is strange at a first sight because it says that this variant saturates BPS bound while being nonsupersymmetric (since the function χ does not vanish in this case). An explanation of the fact is given below in the paragraph 5.4.3.

Variant IIIB where $R = \infty$, $\kappa = 0$ and $\theta = \frac{1}{2\tilde{d}}$ analogously as the variant IB describes a spacetime stretched between two boundaries where the branes sustain nonzero curvature.

ϑ	$-\infty$	0	$\theta = \frac{1}{2\tilde{d}}$	$+\infty$
r	0	1	∞	undefined
$\dot{S}e^{-B_\vartheta}$	finite	finite	finite	finite
δt	∞	finite	finite	∞
$\delta\tau$	finite	finite	∞	finite
e^A	0	finite	∞	0
e^{B_ϑ}	0	finite	∞	0
e^B	$O(r^{d/d-2})$	finite	$O(r^{d/d-2})$	undefined

Table 7: Variant IIIB.

Variant IIIC where $R = \infty$, $\kappa = 0$ and $\theta < \frac{1}{2\tilde{d}}$ is similar to the variant IIIA but now the naked singularity emerges in the area covered by isotropic coordinates.

Variant IVC where $R \in (0, \infty)$, $s_\chi = -1$, $\kappa > 0$ and $\theta \neq \vartheta_0$, where it is defined $\vartheta_0 = -\frac{(1/R)^{\tilde{d}} + R^{\tilde{d}}}{2\tilde{d}} \arctan R^{\tilde{d}}$, then:

$$\rho = \frac{1}{4} \left((1/R)^{\tilde{d}} + R^{\tilde{d}} \right) \left| \sin \left(\frac{4\tilde{d}\vartheta}{(1/R)^{\tilde{d}} + R^{\tilde{d}}} + 2 \arctan R^{\tilde{d}} \right) \right|, \quad (423)$$

$$\omega = \frac{|p|\sqrt{\Delta}}{2\sqrt{\kappa}} |\sin(\sqrt{\kappa}(\vartheta - \theta))|. \quad (424)$$

In this variant the functions ω and ρ are periodic and a interval of ϑ closed between two adjacent zeros of ρ corresponds to $r \in (0, \infty)$. At the both ends given by $\vartheta = \vartheta_0$ ($r = \infty$) and $\vartheta = \vartheta_0 + \frac{\pi((1/R)^{\tilde{d}} + R^{\tilde{d}})}{4\tilde{d}}$ ($r = 0$) the spacetime tends to flat Minkowski spacetime. But because the period of ω is shorter than the period of ρ there is always at least one naked singularity between the infinities. Localization of the singularities is given by $\vartheta = \theta + \frac{\pi}{\sqrt{\kappa}}$.

ϑ	ϑ_0	0	θ	$\vartheta_0 + \frac{\pi((1/R)^d + R^d)}{4\tilde{d}}$
r	0∞	1	finite	0∞
$\dot{S}e^{-B_\vartheta}$	0	finite	∞	0
δt	∞	finite	finite	∞
$\delta\tau$	∞	finite	finite	∞
e^A	finite	finite	∞	finite
e^{B_ϑ}	∞	finite	0	∞
e^B	finite	finite	0	finite

Table 8: Variant IVC.

Variant IVC/B where $R \in (0, \infty)$, $s_\chi = -1$, $\kappa > 0$ and $\theta = -\frac{(1/R)^{\tilde{d}} + R^{\tilde{d}}}{2\tilde{d}} \arctan R^{\tilde{d}}$.

ϑ	$\theta = \vartheta_0$	0	$\theta + \frac{\pi}{\sqrt{\kappa}}$	$\vartheta_0 + \frac{\pi((1/R)^d + R^d)}{4\tilde{d}}$
r	0∞	1	finite	0∞
$\dot{S}e^{-B_\vartheta}$	finite	finite	∞	0
δt	finite	finite	finite	∞
$\delta\tau$	∞	finite	finite	∞
e^A	∞	finite	∞	finite
e^{B_ϑ}	∞	finite	0	∞
e^B	0	finite	0	finite

Table 9: Variant IVC/B.

This variant is a variation of the previous one and describes a situation when θ coincides with ϑ_0 . Then at the point $\vartheta = \theta = \vartheta_0$ curvature of the spacetime is finite and nonzero due to presence of the brane attached at this point. But there is necessarily another brane at $\vartheta = \theta + \frac{\pi}{\sqrt{\kappa}}$ which produces a naked singularity.

Summarizing, we see that the parameter R gives a position of a horizon expressed with the radial isotropic coordinate r . Similarly a point described in ϑ coordinate as $\vartheta = \theta$ can be interpreted as a localization of the brane. In the variants of type A and type C the points $\vartheta = \theta$ are singular.

5.4.3 Supersymmetry.

When $D = 11$ and $d = 3$ the variants reviewed in the previous paragraph can be naturally interpreted as describing the electric 2-brane in the eleven dimensional supergravity. This gives us a possibility to test supersymmetric properties of the variants directly i.e. by examining the formulae (162) and (163). From those we have the following rules:

- $\delta_\eta \Psi_\mu = 0$ only if:

$$e^{N_+} \frac{d}{dr} (e^{3A} + e^C) \eta_{0+} + e^{N_-} \frac{d}{dr} (e^{3A} - e^C) \eta_{0-} = 0, \quad (425)$$

- $\delta_\eta \Psi_m = 0$ only if:

$$\left(N'_+ + \frac{1}{6} e^{C-3A} C' \right) e^{N_+} \eta_{0+} + \left(N'_- - \frac{1}{6} e^{C-3A} C' \right) e^{N_-} \eta_{0-} = 0, \quad (426)$$

$$\left(B' - \frac{1}{6} e^{C-3A} C' \right) e^{N_+} \eta_{0+} + \left(B' + \frac{1}{6} e^{C-3A} C' \right) e^{N_-} \eta_{0-} = 0, \quad (427)$$

where we decomposed the parameter η with respect to eigenstates of the operator Γ_V (159):

$$\eta = e^{N_+} \eta_{0+} + e^{N_-} \eta_{0-}, \quad (428)$$

where

$$\Gamma_V \eta_{0\pm} = \pm \sigma \eta_{0\pm} \quad (429)$$

and $\eta_{0\pm}$ are constant spinors and N_\pm are functions dependent on the radial coordinate.

We can always choose such N_+ and N_- which satisfy (426) identically for arbitrary η_{0+} and η_{0-} . In the group of supersymmetry parameters η these N_+ and N_- define a subgroup which is possibly preserved by the solution. So the preserved supersymmetry is necessary rigid with respect to the coordinate r . Examining the other conditions we verify if the supersymmetry is really preserved.

For the variant I we find that:

$$\delta_\eta \Psi_\mu = \frac{1}{3} e^{N_+} \Gamma_\mu \Gamma^m \frac{y_m}{r} (e^{3A})' \eta_{0+}, \quad (430)$$

$$\delta_\eta \Psi_m = \frac{1}{4} \Gamma_m{}^n \frac{y_n}{r} (e^{-\frac{1}{2}A})' \eta_{0+}, \quad (431)$$

so this part of supersymmetry which is described by η_{0+} is always preserved. Similarly for the variant III we have:

$$\delta_\eta \Psi_\mu = \frac{1}{3} e^{N_+} \Gamma_\mu \Gamma^m \frac{y_m}{r} (e^{3A})' \eta_{0+}, \quad (432)$$

$$\delta_\eta \Psi_m = \frac{1}{4} \Gamma_m{}^n \frac{y_n}{r} (e^{+\frac{1}{2}A})' \eta_{0-}, \quad (433)$$

The first condition is the same as for the variant I, but the second is changed replacing η_{0+} with η_{0-} . Consequently no supersymmetry can be preserved.

It is a very important fact that the supersymmetry transformations of the gravitino components with the vector index corresponding to the directions tangent to the brane (432) break other part of supersymmetry than the components which have the index related to the transversal directions (433). So, considering a model dimensionally reduced only to the subspace parallel to the brane or only to the subspace orthogonal to the brane we can see only a term breaking half of supersymmetry. This explains why for the nonsupersymmetric variant III we have the same relation between the charge and the energy density as for the supersymmetric variant I. Calculating the densities we conduct only an integration over the transversal directions, so we in fact work only with the subspace orthogonal to the brane. But in the subspace the η_{0+} part of supersymmetry is still preserved and the BPS inequality has to be saturated.

For variants II and IV analogs of the both conditions (430–431) possess terms proportional to η_{0+} and η_{0-} , so supersymmetry is broken in both tangent and transversal subspaces separately.

5.4.4 Solution without dilaton for $\tilde{d} = -1$.

It was already noted that the case of $\tilde{d} = -1$ is specific because then all the invariants (387 – 389) of the metric tensor are constant finite and nonzero numbers. However, we shoul note that the conditions defining the considered case are rather unphysical, because there is no known supergravity theory without any dilaton but with an antisymmetric field supporting a $(D - 2)$ -brane. So we can treat the case considered here only as a toy model helping to understand properties of some more realistic brane configurations with $\tilde{d} = -1$. See for example the one studied in the paragraph 5.5.2.

As was already mentioned because of the assumptions made we should not expect any naked singularity in the solution. See also that (381) and (382) take forms:

$$e^A = \omega^{-\frac{1}{d}}, \quad (434)$$

$$e^{B_\theta} = 2e^{-\epsilon_X} \omega^{-1}, \quad (435)$$

so the solution expressed in terms of ϑ does not have any dependence on the function ρ . Consequently (403 – 404) can be rewritten as:

$$\delta\tau = \int_{\vartheta_1}^{\vartheta_0} \frac{1}{E} \omega^{-1-\frac{1}{d}} d\vartheta, \quad (436)$$

$$\delta t = \int_{\vartheta_1}^{\vartheta_0} \omega^{\frac{1}{d}-1} d\vartheta. \quad (437)$$

Moreover if $\tilde{d} = -1$ then not only the constant Λ_c vanishes (as it is for $\tilde{d} > 0$) but we have also $\Lambda_\chi = 0$ and the identity holds for all values of R . It suggests that for the solution we have $\mathcal{E} = |V| \frac{2}{\sqrt{\Delta}} \lambda$. But further analysis shows that it is not true, because to calculate properly the energy density we should add an external contribution originating from the brane in each variant.

It can be proved that for the variants I, II and III there is a coordinate system singularity and an event horizon at $\vartheta = \pm\infty$ where the functions e^A and e^{B_ϑ} vanish. It is possible to identify the points $\vartheta = -\infty$ and $\vartheta = +\infty$, so the areas of large positive and negative values of ϑ are related to spacetime at opposite sides of the horizon. A point where $\vartheta = \theta$ can be reached from the left and the right side by a massless particle in infinite affine parameter but a finite amount of the coordinate time, so it describes boundaries of the spacetime where domain walls – the branes – are localized. The variant IV is different, because then we have no horizon and a series of "infinite" points $\vartheta = \theta + n \frac{\pi}{\sqrt{\kappa}}$. So any interval $\vartheta \in (\theta + \frac{n\pi}{\sqrt{\kappa}}, \theta + \frac{(n+1)\pi}{\sqrt{\kappa}})$ describes a geodesically complete universe between two domain walls.

5.4.5 Solution with dilaton for $\tilde{d} \geq 1$.

Imposing in the diagonal Δ solution constraints $N_A = 1$ and $\tilde{d} \geq 1$ we recover the single brane solution found in [111] and discussed in [112]:

$$e^A = E_A \omega^{-\frac{2\tilde{d}}{(D-2)\Delta}} e^{c_A \vartheta}, \quad (438)$$

$$e^{B_\vartheta} = \left(\frac{1}{2} E_A^{-d} e^{\epsilon_\chi} \right)^{1/\tilde{d}} \omega^{\frac{2d}{(D-2)\Delta}} e^{-\frac{d}{\tilde{d}} c_A \vartheta} \rho^{-1-\frac{1}{d}}, \quad (439)$$

$$e^\phi = E_A^{2\zeta d/a} \omega^{2a/\Delta} e^{-\frac{s2d}{a} c_A \vartheta}, \quad (440)$$

$$e^C = \begin{cases} E_C - \frac{4\sqrt{\kappa}}{p\Delta} \cot(\sqrt{\kappa}(\vartheta - \theta)), & \text{for } \kappa > 0, \\ E_C - \frac{4}{p\Delta}(\vartheta - \theta)^{-1}, & \text{for } \kappa = 0, \\ E_C - \frac{4\sqrt{-\kappa}}{p\Delta} \coth(\sqrt{-\kappa}(\vartheta - \theta)), & \text{for } \kappa < 0, \end{cases} \quad (441)$$

where:

$$\kappa = \frac{\Delta}{2} (\Lambda_\chi + \Lambda_c), \quad (442)$$

$$\Lambda_c = \frac{c_A^2 d}{a^2 \tilde{d}} (D-2)\Delta. \quad (443)$$

It seems that the solution depends on seven parameters: $E_A, C_A, \epsilon_\chi, E_C, \theta, R$ and p . But E_C has no physical meaning and describes gauge freedom of the antisymmetric potential. Other three can be determined if we assume that at a given point $\vartheta = \vartheta_0$ the functions $A = A_{I=\{1\}}$, B and $\phi = \phi_{i=1}$ have a certain value. It is convenient to choose ϑ_0 at a point which is reachable only in infinite affine parameter and where the spacetime is flat and demand:

$$\chi(\vartheta_0) = 0, \quad (444)$$

$$\omega(\vartheta_0) = 1, \quad (445)$$

$$E_A = e^{-c_A \vartheta_0}. \quad (446)$$

Then e^A , e^B and e^ϕ are normalized to one at this point. The constraints (444 – 446) allow then to express e_χ , θ and E_A by the remaining parameters. The remaining three degrees of freedom should then be related to physical quantities: charge, mass and the tachyon condensate [112].

Calculating energy density of the brane, we obtain:

$$\mathcal{E}(A) = \Omega_{\tilde{d}+1} |V| \frac{2}{\sqrt{\Delta}} \sqrt{\lambda^2 - \frac{1}{2} (\Lambda_\chi + \Lambda_c)}. \quad (447)$$

But analogously as it was discussed in the paragraphs 5.4.1 and 5.4.2 it is valid only when certain conditions are fulfilled assuring that the integration has a well defined physical meaning. Comparing to (386) we see that an additional term $-\frac{1}{2}\Lambda_c$ appeared. The term contributes with a nonpositive value (if $\tilde{d} \geq 0$) and depends only on the parameter c_A . This is the reason why we can identify c_A with the tachyon condensate.

Note that nonzero c_A and thus Λ_c is not a direct consequence of a presence of the dilaton in the considered model. See that a total number of constants c_I, c_α and c_B is equal to $N_g + N_\phi$ and there are $N_A + 1$ constraints imposed on them (294) and (296). So we should expect similar effect in any situation when a total number of functions A_I and ϕ_α in the model exceeds the number of branes N_A .

5.4.6 $r \rightarrow 1/r$ duality.

Analyzing the formula (260) giving the definition of the coordinate ϑ we can see that it exhibits the duality:

$$\vartheta(r; R, s_\chi) = -\vartheta(1/r; 1/R, s_\chi), \quad (448)$$

what establishes a relation between ϑ defined for different values of the parameter R . The same transformation applied to the function ρ (269) leads to:

$$\rho(\vartheta; R, s_\chi) = \rho(-\vartheta; 1/R, s_\chi). \quad (449)$$

Further, having a particular solution, for example (378) one can extend the duality to all parameters appearing in the solution:

$$\begin{pmatrix} \vartheta; \theta_i, \kappa_i \\ R, s_\chi, \epsilon_\chi \\ p_i, \sigma_i \\ c_I, c_B, c_\alpha \\ E_I, E_B, E_\alpha \end{pmatrix} \rightarrow \begin{pmatrix} -\vartheta; -\theta_i, \kappa_i \\ 1/R, s_\chi, \epsilon_\chi \\ -p_i, -\sigma_i \\ -c_I, -c_B, -c_\alpha \\ E_I, E_B, E_\alpha \end{pmatrix} \quad (450)$$

and check that the transformation is a symmetry of the fields $e^{A_I}, e^{B_\vartheta}, e^{\phi_\alpha}$ and e^{C_i} (287–289, 380). The function e^B is not invariant under (450), but $e^B dr$ is.

One can be afraid that the duality is only a mathematical trick and both the solutions tied by the duality are physically equivalent up to reversing sing of the brane charge. However, it can be checked that it is not true. To see that it is enough to look at the function χ . Under (450) one has:

$$\chi(1/r; 1/R, s_\chi, \epsilon_\chi) = \chi(r; R, s_\chi, \epsilon_\chi) + 2\tilde{d} \ln r. \quad (451)$$

This gives in particular that the supersymmetric solution given by $R = 0$ (variant I) has a non-supersymmetric partner which can be described by $R = \infty$ (variant III).

From (450) it follows that while the metric tensor and the dilaton are unchanged under the duality, the antisymmetric tensor changes its sign. However our knowledge about the antisymmetric tensor is limited to only those components of the field which spun the brane. Even less we know about fermions. Because the considered model is restricted only to the bosonic truncation of some supergravity theory it is difficult to guess how the spinor fields appearing in the original theory behave under the transformation. But we can collect some information by comparing properties of the supersymmetry transformations (430) and (431) with (432) and (433). This suggests that the duality (450) has a different effect when acting on the components Ψ_μ and Ψ_m .

5.5 Composite branes.

Let us assume now that all except one condition imposed to construction of the model discussed in the paragraph 5.1.1 are still valid. Therefore we allow branes to be composite now. We should recall that validity of the intersecting branes solution derived in the previous model (where all branes are supported by different fields) can be extended to the case of composite branes if the nondiagonal elements of the stress-energy tensor $T(A)_{MN}$ (151) for each antisymmetric field vanish i.e. identity (189) is satisfied. With results developed in [116] we see that the tensor $T(A)_{MN}$ can (but does not have to) be nondiagonal only if at least one of the following situations occur:

1. There are two electric branes with worldvolumes V_1 and V_2 such that:

$$\dim(V_1 \cap V_2) = \dim V_1 - 1. \quad (452)$$

Of course $\dim V_1 = \dim V_2$.

2. There are two magnetic branes with worldvolumes V_1 and V_2 such that:

$$\dim(V_1 \cap V_2) = \dim V_1 - 1. \quad (453)$$

In this case also $\dim V_1 = \dim V_2$.

3. There is an electric brane with worldvolume V_e and a magnetic with V_m such that:

$$\dim(V_e \cap V_m) = 0 \quad \text{and} \quad \tilde{d} = -1. \quad (454)$$

4. There is an electric brane with worldvolume V_e and a magnetic with V_m such that:

$$\dim(V_e \cap V_m) = 0 \quad \text{and} \quad \tilde{d} = 0. \quad (455)$$

5. There is an electric brane with worldvolume V_e and a magnetic with V_m such that:

$$\dim(V_e \cap V_m) = 1 \quad \text{and} \quad \tilde{d} = 0. \quad (456)$$

6. There is an electric brane with worldvolume V_e and a magnetic with V_m such that:

$$\dim(V_e \cap V_m) = 1 \quad \text{and} \quad \tilde{d} = 1. \quad (457)$$

5.5.1 Intersecting composite branes of $D = 11$ supergravity.

Let us focus on the eleven dimensional supergravity theory now. The theory contains only one antisymmetric field, so if one wants to consider commonly orthogonally intersecting branes in the theory they have to be composite. The previously developed model can be employed to give a description of the configuration only if the branes do not fall into classes characterized by (452 – 457) and satisfy all other conditions presented in paragraph 5.1.1. But even that it is not sufficient yet.

The $D = 11$ supergravity possesses the Chern-Simons term $F \wedge F \wedge A$ (5) which was neglected in our model, so to be precise instead of $a = 0$ version of the equation (207):

$$\nabla_M F^{MN_1 N_2 N_3} = 0, \quad (458)$$

we should discuss in the theory:

$$\nabla_M F^{MN_1 N_2 N_3} + \frac{1}{2(4!)^2} \epsilon^{N_1 \dots N_{11}} F_{N_4 \dots N_7} F_{N_8 \dots N_{11}} = 0. \quad (459)$$

Fortunately it can be checked that if the branes are commonly orthogonally intersecting, delocalized (i.e. with the solution depending only on a radial coordinate in the overall transverse space) and with

no.	D_1	D_2	$d_{\{1,2\}}$	$d_{\{1\}}$	$d_{\{2\}}$	$\dim V_\emptyset$	\tilde{d}	diag T
two magnetic 5-branes								
1.	6	6	5	1	1	4	2	2.
2.	6	6	4	2	2	3	1	y
3.	6	6	3	3	3	2	0	y
4.	6	6	2	4	4	1	-1	y
a magnetic 5-brane and an electric 2-brane								
5.	6	3	3	3	0	5	3	y
6.	6	3	2	4	1	4	2	y
7.	6	3	1	5	2	3	1	6.
two electric 2-branes								
8.	3	3	2	1	1	7	5	1.
9.	3	3	1	2	2	6	4	y

Table 10: Two intersecting branes.

timelike worldvolumes, a set of configurations with nonzero contribution from the Chern-Simons term is very limited. We can describe the configurations as containing at least one electric brane and at least two magnetic branes such that:

$$V_e = V_{m,1} \cap V_{m,2} \quad \text{and} \quad \tilde{d} = 0, \quad (460)$$

where V_e denotes the electric brane worldvolume and $V_{m,1}, V_{m,2}$ worldvolumes of the magnetic branes.

Let us make a brief scan of possible two-brane and three-brane intersections in the eleven dimensional supergravity. There are nine inequivalent two-brane configurations which fall into three categories:

- two electric branes, there are two configurations of this kind, one with the tensor $T(A)_{MN}$ diagonal,
- electric and magnetic brane – three cases, two with diagonal $T(A)_{MN}$.
- two magnetic branes – four configurations including three described by diagonal $T(A)_{MN}$.

All of them are shown in the table 10. The letter "y" in the last column of the table tells that the tensor $T(A)_{MN}$ is diagonal. And the numbers in the column indicate which of the reason given in at the beginning of the section makes the tensor possibly nondiagonal. The same convention is used in the tables 11, 12, 13 and 14.

The table 11 gives twenty inequivalent configurations which can be constructed of three magnetic 5-branes. Ten of the configurations are described by diagonal $T(A)_{MN}$, so the method developed in the previous sections can be applied.

Similarly in the table 12 we have listed all 22 examples of brane configurations containing two magnetic and one electric brane. Again for ten of them the tensor $T(A)_{MN}$ is diagonal, but in the case number 3. we encounter nonvanishing Chern-Simons term so it has to be excluded from the set of configurations relevant for the model discussed before.

The table 13 shows 12 configurations with two electric and one magnetic brane, five of them are characterized by diagonal stress-energy tensor. And in the table 14 there are contained five three-brane configurations made of electric branes exclusively, but only one has diagonal $T(A)_{MN}$.

Note, that for the $D = 11$ supergravity the formula (284) for the components of the matrix Δ can be written in a simplified form:

$$\Delta_{ij} = 4 - 2(\min(D_i, D_j) - D_{ij}), \quad (461)$$

where

$$D_{ij} = \dim(V_i \cap V_j) = \sum_{I:i,j \in I} d_I. \quad (462)$$

no.	$d_{\{1,2,3\}}$	$d_{\{1,2\}}$	$d_{\{2,3\}}$	$d_{\{1,3\}}$	$d_{\{1\}}$	$d_{\{2\}}$	$d_{\{3\}}$	$\dim V_\emptyset$	\tilde{d}	diag T
1.	5	0	0	0	1	1	1	3	1	2.
2.	4	1	1	1	0	0	0	4	2	2.
3.	4	1	1	0	1	0	1	3	1	2.
4.	4	1	0	0	1	1	2	2	0	2.
5.	4	0	0	0	2	2	2	1	-1	y
6.	3	2	1	1	0	0	1	3	1	2.
7.	3	2	1	0	1	0	2	2	0	2.
8.	3	1	1	1	1	1	1	2	0	y
9.	3	2	0	0	1	1	3	1	-1	2.
10.	3	1	1	0	2	1	2	1	-1	y
11.	2	2	2	2	0	0	0	3	1	y
12.	2	3	1	1	0	0	2	2	0	2.
13.	2	2	2	1	1	0	1	2	0	y
14.	2	3	1	0	1	0	3	1	-1	2.
15.	2	2	2	0	2	0	2	1	-1	y
16.	2	2	1	1	1	1	2	1	-1	y
17.	1	3	2	2	0	0	1	2	0	y
18.	1	4	1	1	0	0	3	1	-1	2.
19.	1	3	2	1	1	0	2	1	-1	y
20.	1	2	2	2	1	1	1	1	-1	y

Table 11: Three intersecting magnetic 5-branes. $D_1 = D_2 = D_3 = 6$.

no.	$d_{\{1,2,3\}}$	$d_{\{1,2\}}$	$d_{\{2,3\}}$	$d_{\{1,3\}}$	$d_{\{1\}}$	$d_{\{2\}}$	$d_{\{3\}}$	$\dim V_\emptyset$	\tilde{d}	diag T
1.	3	2	0	0	1	1	0	4	2	2.
2.	3	1	0	0	2	2	0	3	1	y
3.	3	0	0	0	3	3	0	2	0	y
4.	2	3	1	0	1	0	0	4	2	2.
5.	2	3	0	0	1	1	1	3	1	2.
6.	2	2	1	0	2	1	0	3	1	y
7.	2	2	0	0	2	2	1	2	0	y
8.	2	1	1	0	3	2	0	2	0	y
9.	2	1	0	0	3	3	1	1	-1	y
10.	2	0	1	0	4	3	0	1	-1	y
11.	1	4	1	1	0	0	0	4	2	2.
12.	1	4	1	0	1	0	1	3	1	2. 6.
13.	1	3	2	0	2	0	0	3	1	6.
14.	1	3	1	1	1	1	0	3	1	y
15.	1	4	0	0	1	1	2	2	0	2. 5.
16.	1	3	1	0	2	1	1	2	0	5.
17.	1	2	2	0	3	1	0	2	0	5.
18.	1	2	1	1	2	2	0	2	0	y
19.	1	3	0	0	2	2	2	1	-1	y
20.	1	2	1	0	3	2	1	1	-1	y
21.	1	1	2	0	4	2	0	1	-1	y
22.	1	1	1	1	3	3	0	1	-1	y

Table 12: Intersecting two magnetic 5-branes and an electric 2-brane. $D_1 = D_2 = 6$, $D_3 = 3$.

no.	$d_{\{1,2,3\}}$	$d_{\{1,2\}}$	$d_{\{2,3\}}$	$d_{\{1,3\}}$	$d_{\{1\}}$	$d_{\{2\}}$	$d_{\{3\}}$	$\dim V_\emptyset$	\tilde{d}	diag T
1.	2	1	0	1	2	0	0	5	3	1.
2.	2	1	0	0	3	0	1	4	2	1.
3.	2	0	0	0	4	1	1	3	1	1.
4.	1	2	0	2	1	0	0	5	3	y
5.	1	2	0	1	2	0	1	4	2	y
6.	1	1	1	1	3	0	0	4	2	1.
7.	1	2	0	0	3	0	2	3	1	6.
8.	1	1	1	0	4	0	1	3	1	1. 6.
9.	1	1	0	1	3	1	1	3	1	y
10.	1	1	0	0	4	1	2	2	0	y
11.	1	0	1	0	5	1	1	2	0	1. 5.
12.	1	0	0	0	5	2	2	1	-1	y

Table 13: Intersecting a magnetic 5-brane and two electric 2-branes. $D_1 = 6$, $D_2 = D_3 = 3$.

no.	$d_{\{1,2,3\}}$	$d_{\{1,2\}}$	$d_{\{2,3\}}$	$d_{\{1,3\}}$	$d_{\{1\}}$	$d_{\{2\}}$	$d_{\{3\}}$	$\dim V_\emptyset$	\tilde{d}	diag T
1.	2	0	0	0	1	1	1	6	4	1.
2.	1	1	1	1	0	0	0	7	5	1.
3.	1	1	1	0	1	0	1	6	4	1.
4.	1	1	0	0	1	1	2	5	3	1.
5.	1	0	0	0	2	2	2	4	2	y

Table 14: Three intersecting electric 2-branes. $D_1 = D_2 = D_3 = 3$.

5.5.2 Three magnetic brane solution in $D = 11$ supergravity.

One of the brane intersections in the eleven dimensional supergravity for which we can formulate and next solve the Toda-like equation is the fifth case in the table 11. The configuration consists of three magnetic branes having the four dimensional common intersection and the one dimensional overall transverse space. The case is very interesting because it can serve as a simplified model of the most promising compactification scheme where the $D = 11$ spacetime decomposes like:

$$\mathcal{M}_{11} \rightarrow \mathcal{M}_4 \times \mathcal{K}_6 \times \mathcal{I}, \quad (463)$$

where \mathcal{K} is a Calabi-Yau manifold and \mathcal{I} is a real interval.

In our model we have $\mathcal{M}_4 = V_{\{1,2,3\}}$, $\mathcal{K}_6 = V_{\{1\}} \times V_{\{2\}} \times V_{\{3\}}$ and $\mathcal{I} = V_\emptyset$ what means that $d_{\{1,2,3\}} = 4$, $d_{\{i\}} = 2$ for $i = 1, 2, 3$ and $\tilde{d} = -1$. This gives a diagonal form for the matrix Δ :

$$\Delta = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}. \quad (464)$$

Finally the appropriate solution can be written as:

$$e^A = E \omega_1^{-1/6} \omega_2^{-1/6} \omega_3^{-1/6} e^{c\vartheta}, \quad (465)$$

$$e^{A_{\{1\}}} = E^{-2} \omega_1^{-1/6} \omega_2^{1/3} \omega_3^{1/3} e^{-2c\vartheta}, \quad (466)$$

$$e^{A_{\{2\}}} = E^{-2} \omega_1^{1/3} \omega_2^{-1/6} \omega_3^{1/3} e^{-2c\vartheta}, \quad (467)$$

$$e^{A_{\{3\}}} = E^{-2} \omega_1^{1/3} \omega_2^{1/3} \omega_3^{-1/6} e^{-2c\vartheta}, \quad (468)$$

$$e^{B_\vartheta} = 2e^{\epsilon_X} E^{-8} \omega_1^{1/3} \omega_2^{1/3} \omega_3^{1/3} e^{-8c\vartheta}, \quad (469)$$

where for simplicity A is written instead of $A_{\{1,2,3\}}$ and where ω_i 's are given by (378). Since

$$\Lambda_\chi = 0, \quad \Lambda_c = -36c^2, \quad (470)$$

therefore for the constants κ_i we have:

$$\kappa_1 + \kappa_2 + \kappa_3 = -72c^2. \quad (471)$$

The relation between the energy density and the charges densities is (in those cases when it is well defined):

$$\mathcal{E} = |V| \sum_i \sqrt{\lambda_i^2 - \frac{1}{4}\kappa_i}. \quad (472)$$

This relation suggests that a necessary condition for preserving supersymmetry is $\kappa_i = 0$ and $c = 0$. To verify this it is instructive to consider again when $\delta_\eta \Psi_M = 0$. To discuss the condition we need first to check how the Dirac matrices decompose under (463). We have:

$$\Gamma_M \rightarrow (e^A \Gamma_\mu, e^{A_{\{j\}}} \Gamma_{\mu^{\{j\}}}, e^B \Gamma_{11}), \quad \text{where } j = 1, 2, 3, \quad (473)$$

and:

$$\begin{aligned} \Gamma_\mu &= \gamma_\mu \otimes \gamma_{V_1} \otimes \gamma_{V_2} \otimes \gamma_{V_3}, \\ \Gamma_{\mu^{\{1\}}} &= Id \otimes \gamma_{\mu^{\{1\}}} \otimes \gamma_{V_2} \otimes \gamma_{V_3}, \\ \Gamma_{\mu^{\{2\}}} &= Id \otimes Id \otimes \gamma_{\mu^{\{2\}}} \otimes \gamma_{V_3}, \\ \Gamma_{\mu^{\{3\}}} &= Id \otimes Id \otimes Id \otimes \gamma_{\mu^{\{3\}}}, \\ \Gamma_{11} &= \gamma_V \otimes \gamma_{V_1} \otimes \gamma_{V_2} \otimes \gamma_{V_3}. \end{aligned} \quad (474)$$

The matrices γ_μ are Dirac matrices and $\gamma_V = -i\gamma_1\gamma_2\gamma_3\gamma_4$ the chiral operator in the four dimensional subspacetime $V_{\{1,2,3\}}$. Similarly $\gamma_{\mu^{\{j\}}}$ and γ_{V_j} for $j = 1, 2, 3$ are respectively Dirac matrices and chiral operator on the two dimensional spaces $V_{\{j\}}$. So we can define:

$$\begin{aligned} \Gamma_{V_1} &= i \text{Id} \otimes \gamma_{V_1} \otimes \text{Id} \otimes \text{Id}, \\ \Gamma_{V_2} &= i \text{Id} \otimes \text{Id} \otimes \gamma_{V_2} \otimes \text{Id}, \\ \Gamma_{V_3} &= i \text{Id} \otimes \text{Id} \otimes \text{Id} \otimes \gamma_{V_3}, \end{aligned} \quad (475)$$

which have to satisfy:

$$\Gamma_{V_j} = \frac{1}{2} \epsilon_{\mu^{\{j\}} \nu^{\{j\}}} \Gamma^{\mu^{\{j\}} \nu^{\{j\}}}. \quad (476)$$

Finally from the supersymmetry preserving condition $\delta_\eta \Psi_M = 0$ it follows:

$$\delta_\eta \Psi_\mu = \Gamma_\mu \Gamma^{11} e^{A-B} \left(\frac{1}{2} A' - \frac{1}{12} \sum_i P_i \right) \eta, \quad (477)$$

$$\delta_\eta \Psi_{\mu^{\{k\}}} = \Gamma_{\mu^{\{k\}}} \Gamma^{11} e^{A_{\{k\}}-B} \left(\frac{1}{2} A'_{\{k\}} + \frac{1}{6} \sum_j P_j - \frac{1}{4} P_k \right) \eta, \quad \text{for } k = 1, 2, 3, \quad (478)$$

$$\delta_\eta \Psi_{11} = \left(N' - \frac{1}{12} \sum_j P_j \right) \eta, \quad (479)$$

where we defined projection operators:

$$P_1 = e^{B-2A_{\{2\}}-2A_{\{3\}}} \Gamma^{11} \Gamma^{V_2} \Gamma^{V_3} \lambda_1, \quad (480)$$

$$P_2 = e^{B-2A_{\{3\}}-2A_{\{1\}}} \Gamma^{11} \Gamma^{V_3} \Gamma^{V_1} \lambda_2, \quad (481)$$

$$P_3 = e^{B-2A_{\{1\}}-2A_{\{2\}}} \Gamma^{11} \Gamma^{V_1} \Gamma^{V_2} \lambda_3 \quad (482)$$

and for the parameter η we introduced $\eta(r) = e^{N(r)} \eta_0$ with η_0 – a constant spinor.

From the equation (479) we obtain:

$$N' = 2A' \quad (483)$$

an analog of (167). Next the equations (477 – 478) lead to:

$$c = 0, \quad \kappa_j = 0, \quad (484)$$

for $j = 1, 2, 3$. However, instead of the harmonic gauge condition $\chi' = 0$ we get only:

$$6A' + 2(A_{\{1\}} + A_{\{2\}} + A_{\{3\}})' = 0, \quad (485)$$

$$-2A' - B' = \chi', \quad (486)$$

where χ preserves its general form. This feature is peculiar for all models where the overall transverse space is one dimensional. In such situation all operators Γ_{mn} have to vanish so we do not have any analog of the term $\Gamma_m{}^n (\partial_n B - \frac{1}{6} e^{C-3A} \partial_n C \Gamma_V)$ appearing in (163). Consequently it is not possible from the requirement of supersymmetry to derive any equation involving B' and restore the harmonic gauge condition.

6 Conclusions

The main purpose of this paper was to provide an overview of the way that leads to the notion of branes, to discuss the known supersymmetric brane solutions and to describe new nonsupersymmetric brane solutions. It is generally accepted that the route from the Standard Model through supersymmetric extensions, Grand Unified theories, supergravities to superstrings and M theory is the most promising (albeit extremely difficult in practice) attempt to unify Quantum Field Theory and the General Relativity i.e. two main pillars of contemporary physics. It is a relatively recent result that the theory of strings has much richer spectrum than just perturbative string excitations. The new objects were known for many years in supergravity under the name of branes and (in analogy to instantons in nonperturbative quantum field theory) were solutions of the classical equations of motion. Although we do not have yet a quantum theory of branes we suspect that the elusive 11-dimensional M theory is just a quantum theory of 2-branes (as string theory is perturbatively a quantum theory of 1-branes i.e. strings). Therefore it is important to find and classify as wide class of brane solutions as possible. A special role is played by the presence (or the absence) of supersymmetry. All known string theories are supersymmetric but it does not exclude a possibility that some solutions break (spontaneously) supersymmetry in a similar way as gauge symmetry can be spontaneously broken by the Higgs mechanism. Since supersymmetry is certainly broken at low energy scales it is interesting too search for brane solutions that are nonsupersymmetric to gain some insight into possible ways of supersymmetry breaking in supergravity (and indirectly in string theory).

The branes are multidimensional objects possible to define in many various environments (in the most general situation – on a ground of an arbitrary theory with antisymmetric tensor fields) and provide a very useful method which allows to understand some theories as theories living on brane worldvolumes (or intersection of the worldvolumes) immersed in some other theory. In this scheme we can for example interpret superstring theories as defined on domain walls of the M-theory. Similarly various super-Yang-Mills or nonsupersymmetric Yang-Mills theories can be regarded as superstring theories on respectively BPS or non-BPS configurations made of D- and NS-branes. Finally the Standard Model should appear as a low energy limit of a theory of this kind related to a four dimensional intersection of the brane worldvolumes which break supersymmetry. A verification if it is really possible to derive the Standard Model directly from brane solutions of string theory would give us new insight alternative to the usual route string theory - supergravity - rigid supersymmetry - Standard Model.

The branes appearing in superstrings and M-theory can be described also from supergravity point of view as a special class of solution of equations of motion. Because supergravities are low energy limits of superstring and M-theory and are consistent only at the classical level, in such approach some information especially involved with quantum properties of the branes is lost. But on the other hand branes in supergravities can be studied with the use of exact methods while superstrings provide only perturbative techniques. Therefore results obtained in both the ways are in many cases complementary.

The main part of this work was dedicated to give an exact descriptions of possibly wide class of configurations of the branes in the framework of supergravity. It was shown that for such configuration which can be described as commonly orthogonally intersecting delocalized branes the respective equations of motions reduce to the known Toda-like system i.e. a generalization of the Liouville equation. The reduction to the Toda-like system works both for supersymmetric and nonsupersymmetric cases since it does not depend on imposing on the model a harmonic gauge $\chi = 0$ (a necessary condition for preserving supersymmetry). The key of the reduction is a coordinate change where isotropic radial coordinate r is replaced by a coordinate called ϑ . And the relation $\vartheta(r)$ is a curved space harmonic function with the curvature contribution to the harmonic equation given by $d\chi/dr$.

The resulting Toda-like system is integrable and there are several known classes of the solutions which can be written explicitly with the use of elementary functions. Fortunately even the restricted class of analytically known solutions can be relevant for realistic brane configurations and it is possible to test properties of the quite wide class of brane configurations by studying the exact solutions.

In this paper several examples of such solutions are discussed in much detail to properly interpret supersymmetric properties of the solutions. We examine the single brane solutions with and without a dilaton. The first when $D = 11$ can be treated as the supergravitational description of branes in M-theory and the second (for $D = 10$) as the analogous description of branes in superstring theories. The solutions are in fact families of solutions parametrized by λ giving a charge of the brane and R describing a localization of a horizon expressed in terms of the isotropic coordinate r . The dilatonic solution additionally depends on a parameter c giving the classical description of a tachyon condensate. Also the three magnetic brane configuration in $D = 11$ supergravity possessing a three dimensional (spatial) intersection and one overall transverse direction is studied.

A transformation $r \rightarrow 1/r$ together with $R \rightarrow 1/R$ and $c \rightarrow -c$ gives a duality in the family of the solutions. The parameter $R = 0$ (together with $c = 0$) corresponds to supersymmetric solution when the BPS inequality $\mathcal{E} \geq Q$ is saturated, where \mathcal{E} and Q are respectively energy and charge density of the brane. For $R \neq 0$ supersymmetry is broken. But the solution dual to the supersymmetric one i.e. given by $R = \infty$ although nonsupersymmetric again saturates the BPS bound. The apparent contradiction can be explained by checking that in this case complementary parts of supersymmetry are broken by gravitino components with vector index respectively tangent and transversal to the brane worldvolume. So considering the model as dimensionally reduced only to the directions orthogonal to the brane (or only to the directions parallel) it seems like one part of supersymmetry is preserved. It proves that it is possible to construct a nonsupersymmetric model characterized by properties usually reserved only for supersymmetric ones.

A character of the duality $r \rightarrow 1/r$ is rather unclear in full generality. While it preserves a metric tensor and the dilaton field and reverses sign of antisymmetric tensor one can see (from the form of supersymmetry transformations) that its action on fermions is more complicated (at least flips their chirality). Since we considered only bosonic truncated model we cannot verify this fact - the model should be extended to incorporate fermions with possibly nonzero vacuum values. Such an extension is rather natural with nonsupersymmetric solutions because preserving supersymmetry condition immediately leads to $\Psi_M = 0$ while there is no analogous condition in the nonsupersymmetric case.

Other generalizations of the model given in this paper can be obtained when one discusses intersections at angles instead of the orthogonal ones or localized branes instead of the delocalized ones. It would be very interesting to find exact solutions in this cases and check if any of them can serve as backgrounds for realistic compactification models.

The possibility of finding the exact nonsupersymmetric solutions is very interesting. The solutions can be used to verify various mechanisms postulated to break supersymmetry in a way leading to the Standard Model. With the general solution at hand one is able to check if conditions under which the mechanisms work can be consistently derived from the equations of motion of the underlying theory. In any realistic case however, it is necessary to go beyond the bosonic solutions of the equations of motion discussed in this paper and include fermions if one wants to take into account fermionic condensates, calculate masses, chiralities, coupling constants etc. and compare it to the Standard Model or its extensions with broken supersymmetry.

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